Alternating Current RL Circuits

1 Objectives

1. To understand the voltage/current phase behavior of RL circuits under applied alternating current voltages, and
2. To understand the current amplitude behavior of RL circuits under applied alternating current voltages.

2 Introduction

You have studied the behavior of RC circuits under both direct and alternating current conditions. The final passive component we must consider is the *inductor*. The voltage across a capacitor is proportional to the charge *on* it ($V(t) = q(t)/C$), and the voltage across a resistor is proportional to the rate at which charge flows *through* it ($V_R(t) = R\mathrm{d}q(t)/\mathrm{d}t$), while the voltage across the inductor is proportional to the *rate of change* of the current through it

$$V_L(t) = -L\frac{\mathrm{d}I(t)}{\mathrm{d}t} \ .$$

The minus sign indicates that the voltage across the inductor seeks to counter the changing current (a phenomenon known as Lenz's Law).

In a previous lab[1] you studied the behavior of the RC circuit under alternating applied (or AC) voltages. Here, you will study the behavior of a similar circuit where the capacitor is replaced with an *inductor*; see Figure 1.

The SI unit of inductance is the henry, with symbol H:

$$\mathrm{H} = \Omega\,\mathrm{s} = \frac{\mathrm{kg}\,\mathrm{m}^2}{\mathrm{C}^2} \ .$$

[1] *Alternating Current RC Circuits*

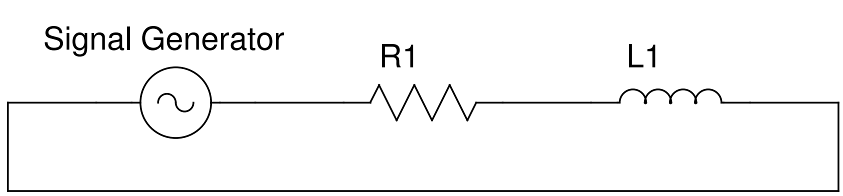


Figure 1: The RL circuit.

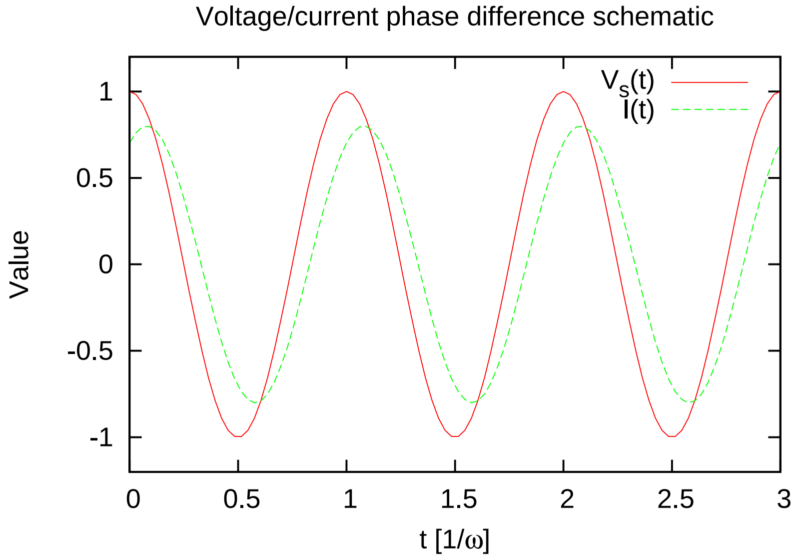


Figure 2: A schematic of the phase difference between the applied voltage $V(t)$ and the derived current $I(t)$.

The henry is named for Joseph Henry, an 18^{th} century American contemporary of Michael Faraday. Both men discovered electromagnetic induction independently and contemporaneously. Since Faraday is honored in the unit of capacitance, Henry is honored with the unit of inductance.

3 Theory

Once again, let's analyze this circuit using Kirchoff's Rules. As always, you find

$$V_s(t) - V_R(t) - V_L(t) = 0 \ ,$$

leading to the differential equation

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} + \frac{R}{L} I(t) = \frac{V_s(t)}{L} \ .$$

The behavior of this *current* is, in all important ways, *identical* to the behavior of the *charge* on a capacitor. The time constant for current changes is $\tau = L/R$. Again, we solve the equation in all its particulars in Appendix A. For a sinusoidally varying source voltage

$$V_s(t) = V_s \cos \omega t \ ,$$

we find the current is again out of phase, but this time it *lags* the voltage (see Figure 2)

$$
\begin{aligned}
I(t) &= \frac{V_s}{R}\frac{1}{\sqrt{1+(\omega\tau)^2}}\cos(\omega t-\phi) \\
V_R(t) &= V_s\frac{1}{\sqrt{1+(\omega\tau)^2}}\cos(\omega t-\phi) \\
V_L(t) &= V_s\frac{\omega\tau}{\sqrt{1+(\omega\tau)^2}}\sin(\omega t-\phi)\ ,
\end{aligned}
$$

where the phase is given by

$$\tan\phi = \frac{\omega L}{R}\ .$$

Just as for the capacitor circuit, we can define and use the concept of *impedance* to predict the behavior of the circuit. The combination ωL has units of resistance, and we define the *inductive reactance* by

$$X_L = \omega L\ .$$

The phase is given by

$$\tan\phi = \frac{X_L}{R}\ ,$$

while the circuit impedance is given by

$$Z^2 = R^2 + X_L^2\ ,$$

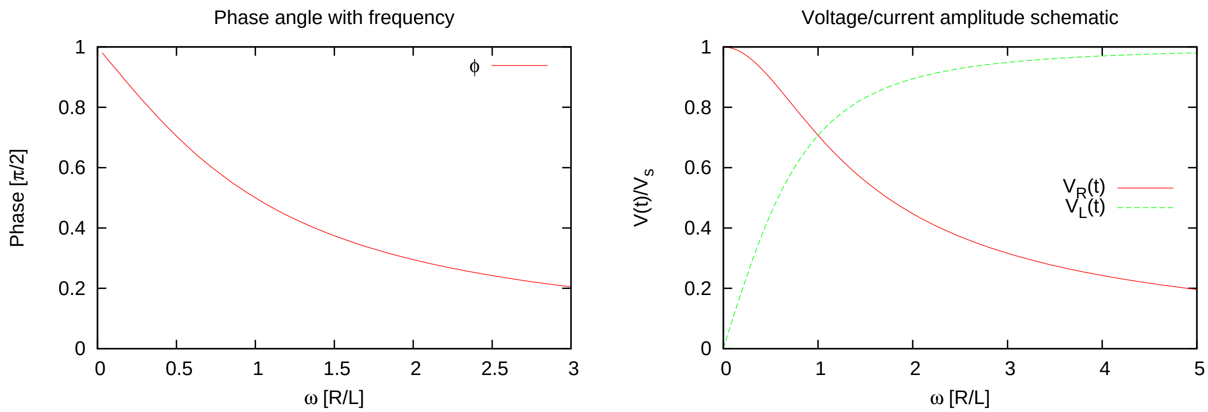
which has all the same consequences for the voltage amplitudes as it did for the RC circuit.

Just as we did for the RC circuit, we can consider the behavior of this circuit as a function of frequency. In the limit that the frequency goes to zero, the current will be steady state. Since steady state means "no change", the voltage across the inductor must vanish, and the phase has to go to zero. This is what we see in Figure 3. In the other extreme, with very high frequencies, the current is changing very quickly, so the voltage will all be visible across the inductor; again, this is what we see in Figure 3. Inductors become transparent at low frequencies, and opaque at high frequencies, just the opposite of the capacitor! Compare the amplitude and phase response curves in Figure 3 with the same curves in the lab *Alternating Current RC Circuits*.

If you are left wondering what happens when you put a AC voltage source, a resistor, a capacitor, *and* an inductor into a single circuit, you're thinking along the right lines. Stay tuned for next week.

4 Procedures

You should receive two multimeters, an oscilloscope, a function generator, a decade resistance box, and a decade inductor box.



(a) Phases (b) Voltage Amplitudes

Figure 3: The phase angle as a function of angular frequency is on the left, while the voltage amplitudes are displayed on the right. In both cases, the frequency is normalized in units of L/R. The phase is normalized to $\pi/2$, while the amplitudes are normalized to V_s.

1. First, let's select component values for testing. Choose an inductor value somewhere around 7 H. Select a frequency between 300 Hz and 600 Hz. Calculate X_L and choose a value for $R \approx 1.2X_L$. Measure and record the value of R.

2. Configure the circuit for testing as shown in Figure 1. Insert one of the multimeters to record the AC current; except in the last two steps of the procedure, make sure the current remains constant throughout the experiment.

3. Using the other multimeter, record the frequency f, and the RMS AC voltages across the signal generator V_s, the resistor V_R, and the inductor V_L.

4. Let's measure the phase shift between the current and applied voltage. Connect the oscilloscope so as to measure the voltage across the resistor and signal generator; make sure the negtive inputs share a common reference point. Make sure the two signal baselines are centered with respect to the horizontal and vertical axes of the oscilloscope, and adjust the voltage and time scales so that slightly more than one cycle of both waveforms is visible. Measure the phase shift as you did in the previous lab. Increase the frequency by 50%, and determine the phase shift again. Double the initial frequency, and repeat.

5. Next, map out the amplitude of the current response. Without changing R, vary the frequency over, say, ten points, and record the frequency, RMS voltage V_s and RMS current I at those points. Record you observations of the amplitudes of V_s and V_R on the oscilloscope.

A Derivation of Solutions

The differential equation for the AC RL circuit is given in Section 3, and is nearly identical to the equations we have studied in the DC RC and AC RC cases. In fact, in the DC case, we can just substitute $I(t)$ for $q(t)$ and $\tau = L/R$ for $\tau = 1/RC$, and we have the DC behavior of the RL circuit straightaway. In the discharging case

$$I(t) = I(t_0)e^{-(t-t_0)/\tau} \ ,$$

while in the charging case

$$I(t) = I(t_0)e^{-(t-t_0)/\tau} + \frac{V_s}{R}\left(1 - e^{-(t-t_0)/\tau}\right) \ .$$

Again, like the RC circuit, in the charging RL circuit, the initial current "stored" in the inductor decays away, while the imposed voltage "stores" a current in the inductor exponentially.

In the AC case, very little of the derivation changes between the RC and RL circuits, provided we replace the time constants appropriately. If you follow through all the work from the RC lab, you will find

$$\begin{aligned} I(t) = I(t_0)e^{-(t-t_0)/\tau} - e^{-(t-t_0)/\tau}\frac{V_s}{R}\frac{1}{(1/\tau)^2+\omega^2}\left(\frac{1}{\tau}\cos\omega t_0 + \omega\sin\omega t_0\right) \\ + \frac{V_s}{R}\frac{1}{(1/\tau)^2+\omega^2}\left(\frac{1}{\tau}\cos\omega t + \omega\sin\omega t\right) \ . \end{aligned}$$

Again, the first line is the transient response, while the second line gives the steady state response. Keeping only the steady state response, the same techniques as last time give us the voltage profiles and phases given in Section 3.

Pre-Lab Exercises

Answer these questions as instructed on Blackboard; make sure to submit them before your lab session!

1. Calculate the reactance of a 7 H capacitor at a frequency of 250 Hz.
2. If an RL circuit has a $50\,\Omega$ resistor in series with a 7 mH inductor, what will its impedance be at 500 Hz?
3. An RL circuit has a $5\,\mathrm{k}\Omega$ resistor and a 1 H inductor. At what frequency will the current lag the voltage by $\pi/4$?
4. An RL circuit has a $5\,\mathrm{k}\Omega$ resistor and a 1 mH inductor. This circuit is driven by a 100 Hz sine wave with 1 V amplitude. What is the amplitude of the current in the circuit?

Post-Lab Exercises

1. From your recorded inductance, and measured resistance and frequency, determine the reactance and impedance of your circuit. Make sure to estimate your uncertainties. Determine the impedance experimentally via another method, taking care of the uncertaintites.Do you get the same results?

2. Estimate the uncertainties on the measured values of V_s, V_L, and V_R. Are the three values consistent with each other? Explain what you mean by "consistent".

3. Describe qualitatively what happens to your signals when you vary the frequency.

4. From your measurements in Step 4 of the procedure, determine the phase shift at each of the three measured frequencies, including an estimate of the uncertainty. How do these compare to the theoretical predictions?

5. Is your data from Step 5 consistent with the predictions of theory? Specifically, do the voltage and current amplitudes measured by oscilloscope and by multimeter match, within uncertainties, and do they comport with theoretical expectations?

6. Discuss briefly whether you have met the objectives of the lab exercises.