Contexts, Exercises, Problem Solving and Mathematical Modeling (2019)

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Mathematics often proceeds by introducing technical vocabulary (function, prime number, isosceles, triangle, trapezoid, quadrilateral, etc.) to improve clarity and make sure everyone is talking about the same collection of objects or concept. In mathematics education, when words are used, especially words that are borrowed from general use, it is often helpful to have concrete examples to illustrate the issues of word/concept usage because it is hard to be fully precise. My goal here is to shed light on some of the many terms that surround the teaching of ideas related to mathematical modeling, in order to make "informal" distinctions between such terms as exercises, contexts, open-ended problems, model, problem solving, and modeling.

Students learn by practicing skills and algorithms involved with the tools and concepts they learn about. This practice is carried out by having students do *exercises*. An exercise is typically a type of problem that teachers can create easily that can be used to assess that students have learned a concept or skill.

Exercises:

i. Add: $1/3 + 2/5$

ii. Factor: $x^2 - 4y^2$

iii. Solve for x: 2x - 3(x-2) = x + 4 (check your answer)

iv. Find an Eulerian circuit in the graph below or explain why the graph has no Eulerian circuit:

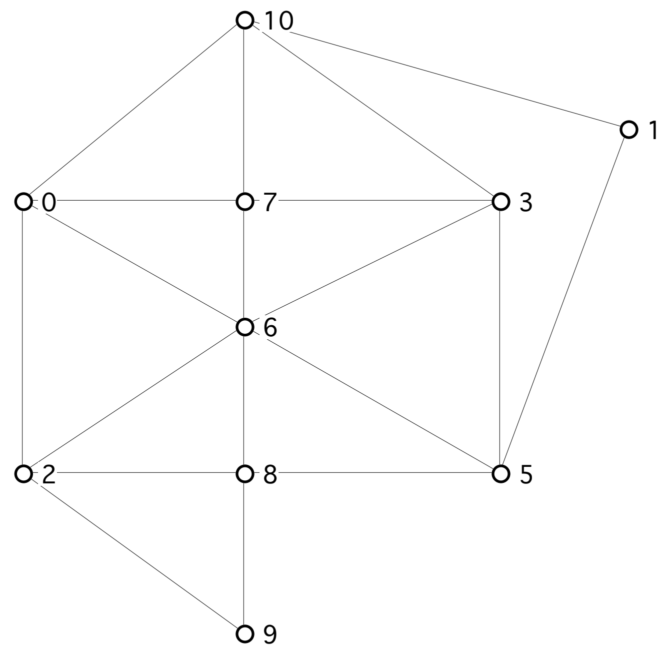

Figure 1

(Note: This diagram has vertex labels that start at 0 and are not consecutive integers. This helps students see that the vertex labels are not being used as counting numbers but only as names. Often exercises can promote goals that go beyond mere mechanics.)

One can introduce mathematical ideas in *contexts*. This refers to finding applied ("real world") settings in which the mathematical concepts, algorithms or processes arise. Thus, instead of talking about the fraction 2/3, one can talk about 2/3 of a pizza or 2/3 of a jar of jam. Instead of saying find an Eulerian circuit in the graph above (Figure 1), one could talk about designing a pothole inspection route (if possible) that starts at vertex 5 in Figure 1 which traverses each street segment once and only once and returns to the vertex where the inspection started. Having raised the idea that edge traversals in graphs can be applied to pothole inspection routes, one's students can be asked to produce similar settings where edge traversals are involved. Common responses are garbage collection, snow removal, or newspaper and advertising flyer routes. When contexts are used there is often no single correct answer (Figure 1 has many different Eulerian circuits) and discussions often are "open ended." Exercises which have a single answer have their place, but so do exercises which have more than one answer. Geometry questions (proofs, for example) can often be attacked and solved from different perspectives. Because approaches to algebra problems are often very algorithmic (e.g. find the solution of $2x - 4 = x + 11$; find the solution of $x^2 - 5x+6=0$) it is possible to lose sight of the fact that in other areas of mathematics, (e.g. combinatorics, graph theory, geometry, probability) there are many ways to attack problems.

Some teachers refer to assigning exercises involving contexts as "word problems." Unfortunately, this phrase comes with lots of negative baggage because for generations teachers have used word problems of the following kind:

Mary was 3 times as old as James was when he was .... How many legs does Mary's cat have? (Ok. This is a "joke," but conveys a serious issue.)

In my opinion problems of this kind are silly and help paint mathematics and the applicability of mathematics in a very poor light. This type of problem should be avoided, and I would banish the term "word problems" in favor of phrases like "problems that involve contexts" or "applications setting " problems.

While exercises such as multiply (x+2) by (x-3) have their place, it is hyperbole to refer to students' doing these kinds of problems as "problem solving." At its core mathematics is about problem solving and theorems. Theorems are new mathematical "facts." Problem solving comes up in exploring ideas, new concepts, applications, and problems that come up on the job in various careers. However, often mathematics educators use problem solving to mean the explorations students do when they are asked about things that they are learning for the first time or that lead to the proofs of theorems. So one talks about developing problem solving skills. These are skills where one is faced with a fairly specific question that one has not seen the likes of before.

George Polya (a mathematician) and Alan Schoenfeld (a mathematician turned mathematics educator) have done lots of work to provide insight into problem solving and problem solving techniques. Among the well known things that are taught as ways to try to help students become better problem solvers are:

a. Restate the problem in different words.

b. List what you are given and what you are trying to find out.

c. Try a simple case.

d. Draw a diagram.

e. Break the problem down into parts and work on the parts separately (if possible).

Many others!

Here is an example of a problem solving task that has been placed in an applications context.

There are 8 middle schools in a large suburban community. The plan is to design a round robin tournament between the girls soccer teams of the schools. (Round robin tournaments involve having each team play exactly once against each other team.) Design an efficient schedule for carrying out the games.

Here is an example of a problem solving task that involves more traditional topics in the high school curriculum but without a context:

When does the polynomial $x^2 + bx + c = 0$ have real roots? When does it have integer roots? One might try to warm up for this question by seeing when $x + a = 0$ has a real root and when it has an integer root.

Now let us look at the issue of the use of the words model, modeling, and mathematical modeling. The basic idea behind a model is that one represents something complex with something simpler. Dolls are used as models of people for children. Companies build models of planes and boats to test in wind tunnels or water tanks to help make sure that design concepts work before trying out the ideas on a much more expensive prototype. The term modeling, as used in mathematics education, is the process of creating a mathematical model. It refers to taking a situation in the "real world," making simplifying assumptions about this situation and using mathematical tools to get insight into this situation. One might model how to design a voting system for a new school club; schedule the matches of the tennis teams at the local high schools in a small town; help a large business decide what product mix to manufacture. The tools one uses are all the mathematical tools one has at one's disposal. One might use equations, differential or partial differential equations, statistical tools, probability theory, matrices, graph theory, linear programming, etc. The more mathematical techniques one knows the more flexibility one has to try to obtain an insightful model. The modeling that can be done by high school students is different from what can be done by graduate students. However, it is often surprising how much insight can be obtained from very simple mathematical models.

Mathematical modeling is often used to mean the field within mathematics which is designed to teach and research the tools for constructing mathematical models. It begins with the construction of some version of the modeling cycle diagram shown in Figure 2.

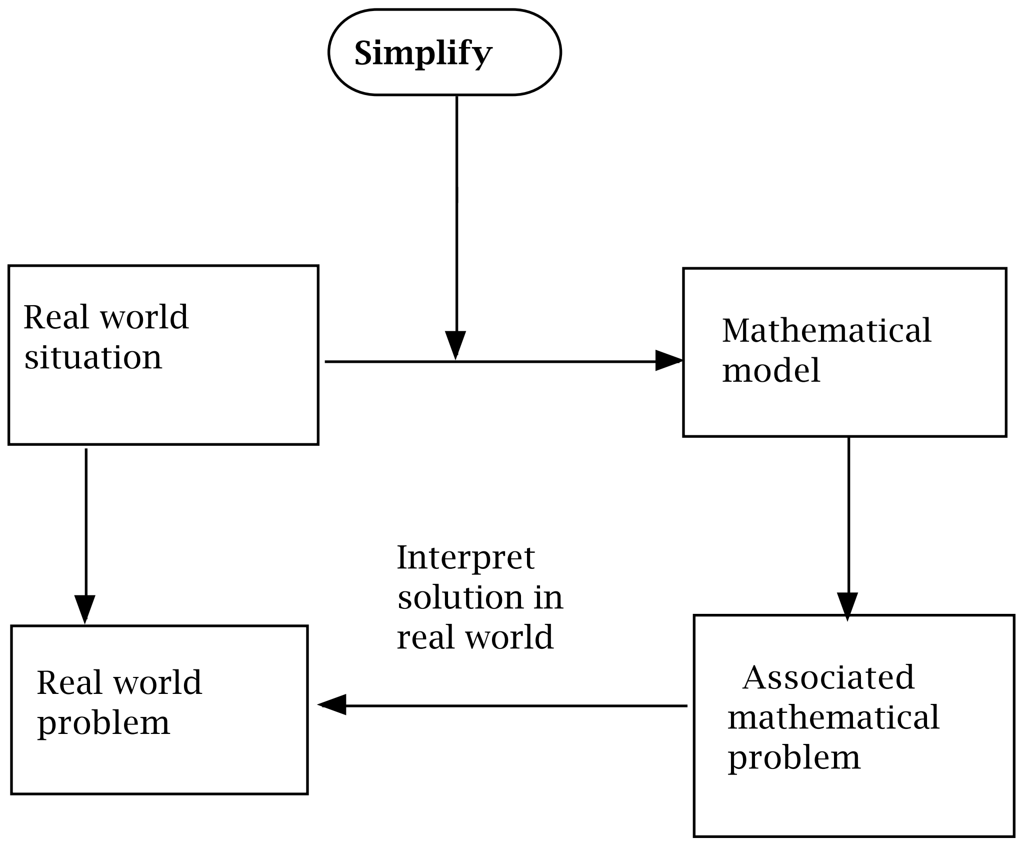

Figure 2

There are two "extreme" approaches to the teaching of mathematical modeling. One starts with a mathematical skill or tool, teaches students how to use this tool, and then gives examples of real world situations where this tool comes into play. For example, one might teach students about linear functions and equations and then show them how to use a linear function to model "break-even" problems in business or find a regression line for some data. The other approach is to start with a motivating real world situation and describe the mathematical tools, some of which may not be familiar to one's students, that have found use in this environment. Thus, one may teach students game theory ideas to attack a fairness question or show them that

Markov chains have value in modeling the sociological question of class mobility. The more mathematics one knows the more "artillery" one can bring to bear on trying to find a model that gives insight.

Comparison example

Here I will consider what I think is a very appealing context and discuss the context briefly and then show examples of exercise, problem solving, and modeling tasks related to this context.

The context involved is that of bankruptcy problems. A bankruptcy occurs when there are creditors who are owed money by a company but that company does not have the cash to pay its bills.

As you consider what follows note that bankruptcy questions are of interest to and related to a variety of similar situations. The key issue is that there is demand for something but not enough "resources" (not necessarily money) to meet the demands involved. Examples, are states competing for the limited water in the Colorado River; applicants for disaster funds after a hurricane, flood or tornado; raising a fixed amount of tax money from different income classes; providing food after a famine; an estate where the amounts promised exceed the size of the estate, etc.

Here is a mathematical formulation (mathematical model) of this problem. One has claimants, A, B, C, ... Z finite in number, with monetary claims (say in dollars) which will also be denoted by A, B, C, etc. The amounts of the claims are verified and assumed accurate and the claimants cannot communicate or act in concert. The assets with which to pay off these claims is E; this amount is fixed and is to be paid out by a fair "judge" who will not get any part of E for his/her work.

Many methods have been proposed to solve such bankruptcy claims. Here is a sample of brief descriptions of such methods. More details can be found by consulting some of the articles in the bibliography, especially the work of William Thompson (an economist) who has written extensively on this subject. To avoid extra words the assumption is there are exactly two claimants. Note that two claimants claim A and B from an estate E. Collectively they wind up losing:

$$E - (A + B)$$

Methods:

1. Total equality of gain

Brief description: Give each claimant exactly half of the amount E.

2. Maimonides gain

Brief description: Treat each claimant as equally as possible but do not give a claimant more than he/she claims.

3. Total equality of loss

Brief description: Give each of the two claimants an amount which equalizes their losses.

Comment: To accomplish this it may be necessary for claimants to add to the amount E with their own money!

4. Maimonides loss

Brief description: Give each of the two claimants the amount which equalizes their losses as much as possible.

5. Proportionality (gain)

6. Proportionality (loss)

There are many other methods. What is interesting is that typically these different methods lead to different answers; students often rediscover these methods on their own. One method, developed in the Middle Ages, however, is not commonly rediscovered, though it is based on elementary ideas.

Sample Bankruptcy Problem:

Here is a typical two claimant problem in the bankruptcy environment:

A's claim	B's claim	Estate size E
80	120	180

Exercises can be formulated by asking students to solve this bankruptcy problem using the different methods.

Here is a brief description of the method developed during the MIddle Ages mentioned above, which is known as either the contested garment rule, or the talmudic method.

Claimant A goes to the judge and states: Claimant B is asking for $120, but the estate E is worth $180. Since there are only two claimants, of the $180, $60 must be mine, since B claims only $120. The $60 is known as A's uncontested claim against B. Claimant B can go to the judge with a similar analysis: A claims only $80, so of the $180, $100 must be mine. The $100 is known as B's uncontested claim against A.

The talmudic method gives each claimant his/her uncontested claim, and whatever remains after these amounts are distributed is split equally among the claimants. For the example above, the final amounts given to claimants A and B are, thus: A is given $60 + $20/2 = $70; B is given $100 + $20/2 = $110.

Also notice that given the "exercise" of finding the amounts given to A and B using the equality of loss method requires that the following system of two linear equations in two unknowns be solved, where a and b denote the amounts given to claimants A and B: $a + b = 180$ and $80 - a = 120 - b$.

As an example of a problem solving question in this environment one might ask:

Under what circumstances, if any, do the proportionality of gain methods and the total equality of loss methods yield the same solution for two claimants?

Finally, as an example of a modeling problem, one could ask students to examine what would constitute a fair way to settle the claims against an estate when there are two claimants.

One of the goals of K-12 mathematics is that students should have the skills and tools to use mathematics to get insight into situations that come up in daily life and in their future careers. Teaching mathematical modeling, and as many mathematical tools as possible to construct mathematical models is of growing importance for teachers at all grade levels.

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Also see: GAIMME Report, sponsored by SIAM and COMAP. (Reprinted: 2019)

https://www.siam.org/Publications/Reports/Detail/Guidelines-for-Assessment-and-Instruction-in-Mathematical-Modeling-Education