BANKRUPT

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Super-rich Americans have lots of money but there are relatively few such people compared with the many more people who are "merely rich," or middle-class, or working class. If the government wishes to raise some fixed amount of money, say $2800 billion (which is approximately the amount of money that the United States Government spent in 2007), for the fiscal year 2009, how much of this money should be raised from each of the different "income classes" in our society?

Now consider a seemingly unrelated economic situation: the recent crisis with the value of various investments in mortgage-based securities. The meltdown in the market for such securities has raised concerns about whether or not the public has been adequately protected from investing its money in a potentially risky way that many people did not fully understand. Furthermore, many homeowners have defaulted on their mortgages and there are concerns about a domino effect if medium- and large-sized companies that own mortgage-based securities start to fail. When one company with limited assets is unable to pay off all its creditors, one says the company is bankrupt. However, the company typically has some remaining assets. What is a fair way for the creditors of a bankrupt company to be paid?

One of the powers of mathematics is to unify, generalize, and abstract. We will soon see that although these two economic problems seem to be rather different, we can get an insight into both of them using some quite simple mathematics. This mathematics applies not only to the two cases above but also to the case of an estate where the claims against the estate exceed its assets.

Let us try to formulate a simple mathematical model for what happens when a bankruptcy occurs. Let us denote by E the size of an *estate* or the amount of money (assets) that a firm that has gone "belly up" has available for its creditors. We will think of E as the net amount that a *fair* referee or judge has to distribute to the firms (or heirs) making claims against E. In the case of an estate, it may be that the terms of a will or the claims against an estate are such that they cannot all be met by the amount remaining in it. The essential nature of a bankruptcy is that the size of the *honest* and verified claims against the bankrupt company *exceed* the amount of the assets to be distributed. For convenience we will use upper case letters for the names of the heirs or creditors. Thus, we might have the following very simple case: E = $120, and there are two *claimants*, or heirs, A and B. Again, to be as specific as possible, we will assume that A is owed $50 and B is owed $100. The general principle we wish to adhere to is that we should be *fair*. Our job is to decide: What part of the $120 available should be given to each of A and B?

Before addressing this question let us take a moment to mention that bankruptcy questions are not the only ones where attempts have been made to use mathematical techniques to try to get insight into how to proceed *fairly*. Mathematics has been used to get insight into fair elections (social choice theory); fair voting systems (e.g., when a new country enters the European Union, what is a fair way to vote on issues that confront the European Union?); and fair games (e.g., is the game fair in Table 1, in which the payoff for playing a particular row and column by the two players gives the payoff (r, c) $r to the row player and $c to the column player?).

	Column 1	Column 2
Row I	(8, –8)	(–1, 1)
Row II	(–64, 64)	(8, –8)

TABLE 1.

Example 1

How might one try, if one were a wise "judge," to solve the problem of distributing an estate E of $120 to two claimants, A and B, who are justly owed $50 and $100, respectively?

One time honored approach to equity is to treat different people equally regardless of what other factors might argue that one treat them differently. For example, the states of Idaho and California, despite the fact that they have very different numbers of people, are each given two senators in the U.S. Congress. In one sense this is fair: All states are treated alike. In another sense it is not fair because the states have such disparate populations. We call the notion of treating all claimants with strict equality *entity equity*.

If we apply entity equity to our specific example, we would award $60 to A and $60 to B. In the spirit of mathematical modeling, we might examine whether this seems a reasonable solution. In this case, entity equity does not meet the test of reasonableness, because by giving A $60 we are rewarding A with even more than A requested. However, there is a germ of a principle here that allows us to improve our model. Perhaps we should treat each claimant as equally as possible but never give a claimant more than he/she requests. From a mathematical standpoint we are asking for the solution to a constrained optimization problem. Philosophically, this approach to solving a bankruptcy problem was advocated by scholars of the Middle Ages, notably the Spanish born (Cordova) Jewish philosopher and polymath Moses Maimonides (1135–1204). In addition to his fame for his philosophical works, Maimonides was known as a physician. When the Almohades entered Spain, Maimonides became an exile in North Africa, first in Morocco, and later Egypt, where he became the personal physician to the Grand Vizier and Sultan.

In our Example 1, if we give each claimant $50, they have been treated equally so far, and we have given collectively $100 to them. We cannot continue to give A more money because A claimed only $50; thus, the remaining amount, $120 – $50 – $50 = $20, is given to B, who gets $70. Thus, A gets $50 and B gets $70: This is called the *Maimonides solution*.

Here is a way of visualizing this process geometrically using different numbers.

Example 2

Suppose A is claiming $300, while B is claiming $60 and there is $210 to distribute. Distribute the funds fairly to the two claimants.

We image that we have 210 units of blue fluid that we have to distribute as equally as possible to the two claimants, whose claims of 300 and 60 are indicated by equal width rectangles with heights, 300 and 60, respectively. We start to pour the fluid into the rectangles keeping the level of the fluid as equal as possible, as shown in **Figure 1**.

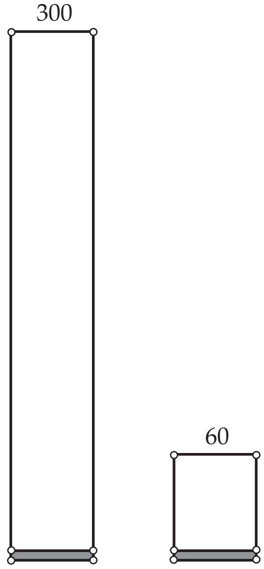

FIGURE 1.

We can keep equalizing the amount given to each of A and B until we reach $60 for each. At this point giving B more will spill over B's vessel, and the distribution looks as in **Figure 2**.

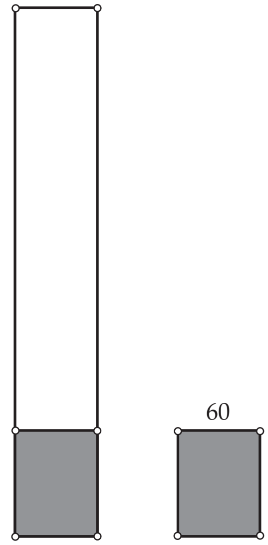

FIGURE 2.

At this stage we have distributed $120: $60 for A and $60 for B. Since we started with $210, we have $90 of additional fluid that we can distribute to A. This means that in the end, A gets $150, while B gets $60.

Visually, we can see what happens in **Figure 3**.

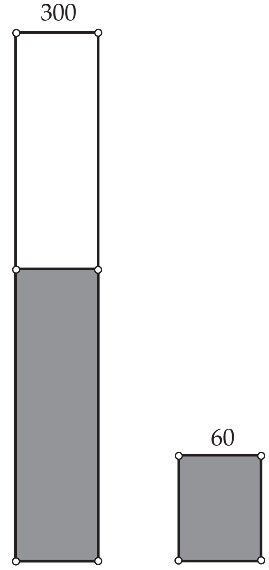

FIGURE 3.

This method applies regardless of the number of claimants. You might want to try your hand at applying Maimonides's method to the following problem:

A's claim 40, B's claim 50, C's claim 150. E = 200.

Also solve for E = 220.

Maimonides's method concentrates on what a claimant gains when his/her claim is *settled*. To be fair, is it what the claimant receives or what the claimant loses that matters? For example, in our original example (Example 1), A receives 50 and loses nothing, while B gains 70 and loses 30. Might it be fairer to try to equalize losses to the two claimants?

Here is one approach to this problem that leads to a problem in algebra, which means that one can use bankruptcy problems as a motivation to solve such algebra problems.

Suppose A is given a and B is given b. We know that $a + b = 120$ and that A's loss is $50 – a$ and B's loss is $100 – b$. If we equalize the claimants' losses, we must have: $50 – a + 100 – b$.

This leads to the system:

$a + b = 120$
$–a + b = 50$

Adding the two equations we get $2b = 170$. Thus, $b = 85$, and we see that $a = 35$. It is easy to check that when A gets \$35 and B gets \$75, the loss for each claimant is \$15. Note that one can avoid using two equations in two unknowns by using algebra to solve this problem, and instead work with one unknown and one equation.

If we do the same type of calculation for our example where A claims 300, B claims 60 and E = 210, we see that to equalize losses, A must get \$225 and B must get –\$15! What does –\$15 mean here? It means that unless B "coughs up" \$15 to A as part of the settlement, losses cannot be equalized. So, if A is given \$225, A's loss is \$75, and if B *subsidizes* the settlement with \$15, B's total loss is \$60 + \$15 or \$75, too!

Can you think of a real world situation where it might be reasonable to ask a claimant to use his/her own money to help subsidize the settlement—so as to equalize loss—as part of solving a bankruptcy?

It seems that this is not fair, but it is the analog of *entity (gain) equality*, and so we might call the solution above *entity (loss) equality*. What should we mean by Maimonides loss in this case? It would mean equalizing loss as much as possible without asking any claimant to subsidize the settlement. One can use a *fluid argument* to do this constrained optimization problem. Verbally, this means that until some money is given to A and/or B, A's loss is \$300 and B's loss is much less, \$60. To try to equalize this requires giving all the \$210 to A, whose loss is largest. This means that A gets \$210 and B gets nothing! A's loss is \$90 and B's loss is \$60. This is as equal as is possible without B subsidizing the settlement.

You may want to try your skill at applying Maimonides's loss method on the examples with three claimants with whom his *gain* method was used above.

A's claim 40, B's claim 50, C's claim 150. E = 200.

Also solve for E = 220.

Having explored four different methods (and two approaches—*gain* and *loss*) to settling bankruptcy problems, let us consider some other appealing points of view to our bankruptcy questions. Since A is owed \$50 and B is owed \$100, the total amount that is owed to them collectively is \$150. We can compute the percentage of the total that each of the claimants is requesting and allot the estate using these percentages.

In this example $\frac{50}{150} = \frac{1}{3}$ so one might argue that A is entitled to $\frac{1}{3}$ of \$120 or \$40, while $\frac{100}{150} = \frac{2}{3}$, so B is entitled to \$80. Note that \$40 and \$80 add to the required \$120 that we are distributing to the claimants. This method is known as *proportionality*. One might put it more precisely: Of the money to be gained, the amounts are proportionally distributed. Is there a different method when one proportionally distributes the losses?

How would that work? The claimants must collectively absorb \$150 – \$120 = \$30 of losses, since there is \$120 to distribute and \$150 in claims. A should absorb a loss of \$10, since $\frac{1}{3}$ of \$30 is \$10, while B should absorb a loss of \$20, since $\frac{2}{3}$ of \$30 is \$20. This means that A gets \$50 – \$10 (A's loss) = \$40, since A is owed \$50, while B gets \$100 – \$20 (B's loss) = \$80, since B is owed \$100.

Note that the assignment that *proportional loss* gives each claimant is the same as *proportional gain*. Is this just an accident or does this always happen? You can try these two methods on Example 2 and see that again the two methods yield the same allocation. Although this is not *a lot* of data, it might be enough to encourage you to try to prove that these two methods always yield the same solution. Using a little algebra, you can verify this easily for the case of two claimants.

Algebra can be used to illuminate another aspect of proportionality. Suppose that the bankruptcy is settled by offering a certain fixed amount of money per dollar to each claimant. For example, suppose each claimant is offered x cents per dollar as a settlement.

We get the equation $50x + 100x = 120$. Solving for x we see that $x = \frac{4}{5}$, or 80 cents per dollar. Since 50(0.80) = \$40 and 100(0.80) = \$80, we get the same solution as the proportional solution. It is not difficult to see that this is always the case and that settling for a fixed number of cents per dollar is equivalent to the proportional solution. Another approach to solving bankruptcy problems is one that I "invented" some years ago. Since E does not cover the amount for which the claimants are asking, one could imagine investing E at a constant interest rate until E grows to the amount equal to the sum of the claims, then distributing to each claimant the amount he/she is owed. Rather than wait for payoff in the future, one could compute the present values of the amounts that the claimants will receive in the future. When one does this, it is not difficult to see that the proportional solution is the amount to give each claimant now!

Are there other methods of resolving bankruptcy questions? Here is a method that might not easily occur to you but is motivated by a very important idea in game theory known as the Shapley value, which is named for the distinguished American game theorist Lloyd Shapley. Although it may seem artificial in this setting, it has interest in situations such as weighted voting games and pervades many aspects of game theory.

To see the full effect, let us consider a three-claimant situation rather than one where there are only two claimants.

We begin by writing down all the permutations of the letters A, B, C, which are the names of the three claimants:

ABC
ACB
BAC
BCA
CAB
CBA

It is easy to see, using elementary ideas from combinatorics (the branch of mathematics concerned with counting problems), that for n claimants there are $n!$ orderings of the names of the claimants.

Now suppose that claimants A, B, and C are owed $30, $90, and $150, respectively, and that the estate E is $120.

Imagine that the judge distributes the $120 to the claimants when they arrive, on a first come first serve basis, but under the assumption that no two claimants arrive together. When the money runs out, it runs out, and if a claimant cannot get his/her full claim because the money was given to claimants who arrived earlier, so be it. For example, if the claimants arrive in the first arrival order above, ABC, the judge gives A $30, B $90 and when C arrives there is nothing left. For the order ACB, the judge gives A $30, C $90 (out of the $120 he/she wants) and B nothing. Continuing in this way we can see what is distributed for all the arrival orders. To find the Shapley value, we average the amounts given to each claimant over all six arrival orders. Because we are taking all arrival orders into account, we are not really rewarding claimants who are faster than others. The results are summarized in **Table 2**.

	A (claim 30)	B (claim 90)	C (claim 150)
ABC	30	90	0
ACB	30	0	90
BAC	30	90	0
BCA	30	90	0
CAB	0	0	120
CBA	0	0	120
Mean	$20	$45	$55

TABLE 2.

For the two-claimant Example 1, we get **Table 3**.

	A (claim 50)	B (claim 100)
AB	50	70
BA	20	100
Mean	35	85

TABLE 3.

Note that in Table 3 the mean is computed by dividing the column sums by 2 since there are 2 arrival orders. Interestingly, the solution we get (A gets $35 and B gets $85) is not one we have seen before.

Table 4 summarizes the different solutions found for Example 1 by adopting different approaches to solving this problem equitably.

E = 120	A's claim is 50; A gets	B's claim is 100; B gets
Entity equity	60	60
Maimonides gain	50	70
Entity loss	35	85
Maimonides loss	35	85
Proportional	40	80
Shapley	35	85

TABLE 4.

It is fun to try to make up an example with two claimants where these different methods yield all different amounts to the claimants A and B.

In the theory of elections, there seems to be no end of appealing methods for deciding a winner for the election, and in bankruptcy questions there are, similarly, as we have seen, many different routes to solutions. However, the next solution we will consider is remarkable for two reasons. First, it grew out of ideas suggested during the Middle Ages, and yet, secondly, people today rarely seem to discover it for themselves. (In contrast, most of the better-known election methods are constantly reinvented.)

The method has sometimes been dubbed the *contested garment* rule, for reasons to be explained below. For now, I will address only the way it can be examined in the bankruptcy context.

Suppose A, noting that B is only claiming $100, goes to the judge and points out that of the $120, the judge must distribute $20 to A since B's claim is only for $100, and there are only two claimants. We will refer to the difference between E and A's claim (assuming it is positive) as A's uncontested claim against B. Note that A's uncontested claim again B is zero if B was claiming an amount equal to or more than the estate E.

Now, does B have an uncontested claim against A? Yes. Since A is only claiming \$50, it follows that of the \$120 there is \$70 that *belong* to B. This uncontested claim against A is zero if the amount A claims is larger than the estate E.

If the judge gives each of the claimants his/her uncontested claims, how much money is left, and what should be done with it? It is not difficult to see, using a bit of algebra, that giving each claimant his/her uncontested claim cannot add to more than the value E, and there must be money left over. What should be done with the leftover money? During the Middle Ages the view was that it should be divided equally among the claimants because they had equal joint claims on this amount.

In Example 1, A would receive his/her uncontested claim of \$20 plus $\frac{\$30}{2}$ (half the amount remaining after the uncontested claims are distributed), giving A \$35. B would receive his/her uncontested claim of \$70 and share half of the amount jointly contested (\$30) for a total of \$85. Note that this, perhaps surprisingly, turns out to be the same as the solution obtained using the Shapley value!

A further insight into the contested garment rule, which has also come to be called the Talmudic method, can be obtained by doing some geometry.

In **Figure 4**, the amount given to A is shown along the x-axis and the amount given to B is shown along the y-axis. The diagram has been scaled so that each unit represents 10 units of payoff. The lines $x = 5$ and $y = 10$ have been drawn to indicate that A should not get more than a payoff of 5 (\$50) and B should not get more than a payoff of 10 (\$100). In the diagram, the points where $x = 5$ and $y = 10$ meet the line $x + y = 12$ are indicated with points A and B, respectively. The segment from A to B constitutes a set of points that game theorists refer to as the *core* of a game. It represents the points that are *individually rational* (that is, no claimant should get more than he/she claims) and *group rational* (that is, collectively the claimants should not get more than the amount available to distribute). In situations where one is trying to solve a bankruptcy with two players, there are always points in the core but many of the games that are hard to solve in a reasonable way involve situations where the core is empty! Each of the methods that we have considered picks out a (usually a different) point of the core.

What is the significance of the point D in the diagram? It is the midpoint of the segment from A to B. The coordinates of D are (3.5, 8.5). Perhaps unexpectedly, this point always gives the contested garment solution to a two-claimant bankruptcy game. (In the two-person case it is also the same as the Shapley value solution.)

The method above is known as the *contested garment rule*, or *Talmudic method*, because it represents the extension of an approach to fair division problems that was described in the Babylonian Talmud. (There is another "Talmud" known as the "Jerusalem Talmud.") The Talmud is a collection of documents that comment on the Old Testament and represent the thinking of Rabbinic scholars and thinkers on the laws and regulations established in the Hebrew Bible. The Babylonian Talmud, whose oral tradition dates to the fifth century, appeared in printed form in the fifteenth century. It includes many attempts to answer practical life questions for people who try to live in accordance with values of the Bible.

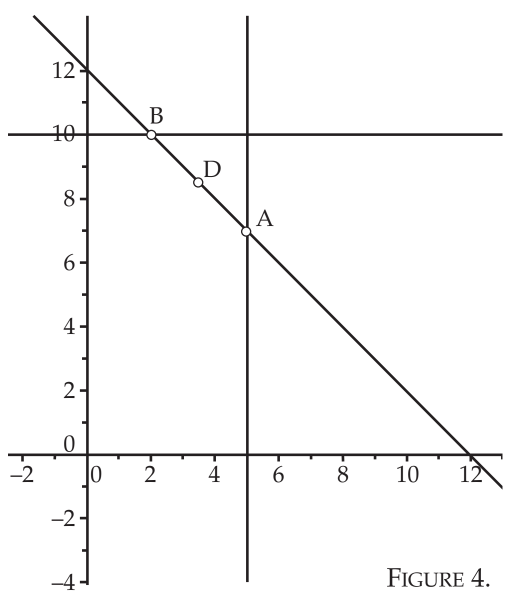

FIGURE 4.

The Babylonian Talmud raised the following question:

Two hold a garment; one claims it all, the other claims half. What is an equitable division of the garment?

The Talmud offers a solution that follows the analysis raised above. The discussion is an ancestor of the modern theory of fair division and games, a remarkable tribute to the thinkers of so long ago. Amazingly, the simple fair division problem raised in Example 1 has given rise to a rich set of mathematical problems and ideas.

(In a sequel, I will discuss how ideas from the Talmud lead to new results in the analysis of bankruptcy games; in particular, how to extend some of the ideas raised above to settling bankruptcy problems with more than two claimants. I will also discuss ideas related to what fairness axioms a "good" bankruptcy problem solution method should obey.) ❏

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