Huntington-Hill:

First row, divide by 0
Second row, divide by $\sqrt{1 \times 2}$
Third row, divide by $\sqrt{2 \times 3}$
Fourth row, divide by $\sqrt{3 \times 4}$

In general $\sqrt{a(a+1)}$ for 0, 1, 2, 3,....
Huntington-Hill is particularly important because it is the method that will be used by LAW to apportion the US House of Representative in after the 2020 Census is finished. Almost certainly NY will again lose seats after this census in Congress. This means required redistricting and perhaps gerrymandering!
Gerrymandering refers to the practice of redrawing districts to achieve some political or racial goal.
What might make one apportionment method better than another one?
Think of the different methods as optimizing some "quantity," and pick the method that does this?
Some notation:

\[ \sum \] means sum; \( h = \text{house size} \)
Population state \( i: p_i \)
\( T = \text{Total population} = \sum p_i \)
Seats assigned to state \( i: a_i \) (integer)
State \( i's \) fair share: \( q_i = ((p_i)/T)h \)
(sometimes called state \( i's \) quota)
Huntington pioneered the use of a different approach to choosing among different apportionments.
His idea was to look at a pair of states and see if society was "happier" with the result when one transferred one seat from one state to the other!
When looking at whether a seat transfer improves things or makes things worse, one needs a way to measure how good things are in the current situation.
One can compare seat transfers between states from either a relative or absolute change point of view.
Reminder:
The relative change between two positive quantities $a$ and $b$ is given by:

$$\frac{|a-b|}{\min (a, b)}$$

while absolute change is given by:

$$|a-b|$$
Two natural ways to think about what a state received in an apportionment are:

representatives per person: 

\[ \frac{a_i}{p_i} \]

people per representative: 

\[ \frac{p_i}{a_i} \]
Now we can measure "disparity" between pairs of states in different ways:
Absolute difference between states j and i:

\[ |\frac{a_j}{p_j} - \frac{a_i}{p_i}| \]

or

\[ |\frac{p_j}{a_j} - \frac{p_i}{a_i}| \]
Perhaps unintuitively, there are usually DIFFERENT apportionments that MUST be chosen if you want to be fair with these different "measures" of fairness, even though $a_j/p_j$ and $p_j/a_j$ are reciprocals of each other!
Webster is the method which is best measured by:

\[|a_j/p_j - a_i/p_i|\]

but when one relativizes this:

\[
\frac{|a_j/p_j - a_i/p_i|}{\min(a_j/p_j, a_i/p_i)}
\]
Huntington-Hill is the best choice!
What are the pros and cons of these different methods? Isn't it somewhat arbitrary which of the 5 rank index (rounding rule equivalents) methods that picks?

Yes and no! But feelings run high.
Fact: If one uses a RELATIVE measure of fairness in comparing two states with regard to who get the next seat, then Huntington proved that Huntington-Hill is mandatory in all cases.
However, in addition to the issue of whether one should measure fairness in absolute or relative terms, there is another important idea to discuss.

Referred to as BIAS.
For a single instance of an apportionment problem many times the 5 methods agree, but sometimes they differ. However, perhaps when the same method is used over and over again for MANY instances there is a systematic way that one method discriminates against states in some way, in particular, due to their population size!
Perhaps one method is BIASED for against SMALL POPULATION states or LARGE POPULATION states.

Looked at in terms of rounding for numbers which lie between consecutive integers a and (a+1) we have this ordering:
Generous in giving an extra seat for a fraction:

Adams
Dean
Huntington-Hill
Webster
Jefferson
There is wide agreement that Adams consistently rewards smaller states with more seats than they "deserve," while Jefferson (D'Hondt) rewards large states with more seats than they "deserve."

Things get heated between what happens for Webster and Huntington-Hill.
The two leading experts on apportionment Balinski and Young argue that Webster is less biased than Huntington-Hill. Others are not sure. In the US apportionment problem the Constitution is biased towards small states, in the sense that one must give each state one seat no matter how small its population!
In general the harmonic mean of $a$ and $a+1$ is smaller than the geometric mean of $a$ and $a+1$, which in turn is smaller than the arithmetic mean of $a$ and $a+1$!
The US Supreme Court has decided a variety of apportionment cases.

In the most important they decided not to agree when Montana went from having 2 seats in the House of Representatives to only 1 seat, that Dean's Method should be used because with Dean's Method Montana would have gotten 2 seats!
Other cases have dealt with the fact that Congress has given over the "mechanics" of carrying out the apportionment to the Commerce Department (which carries out the US Census) and that they use Huntington-Hill.
In the European context of apportionment problems, countries that use D'Hondt method tend to have more "stability" than those that use some of the other methods. Why? The party with the largest vote may get an "assist" by rewarding it with "more seats," which makes it easier to form stable coalitions in parliament. Not all scholars agree!
Other apportionment methods use a different rounding rule:

Jefferson (D'Hondt): Always round down

Adams: Always round up
Huntington/Hill (Currently the method used in America): Round using the geometry mean

Geometric mean of a and b equals square root (ab)
Geometric mean of \(a\) and \(b\):

\[ \sqrt{ab} \]

Geometric mean of \(a\) and \((a+1)\):

\[ \sqrt{a(a + 1)} \]
How does one decide which of these methods is better or worse?

What fairness properties do they obey?
Fairness axioms:

1. House size $h$ monotonicity.
For many practical problems not important and there are algorithms that obey house size monotonicity and other critical fairness properties.
2. Population monotonicity
3. Quota

A state should get its fair share if it is an integer and the fair share rounded up or down to the next integer if it is not an integer.
1,593,436/35 = 45526.743 assigns too many seats.

Table uses: 46842 to assign 35 seats.

<table>
<thead>
<tr>
<th>Party (State)</th>
<th>Vote (population)</th>
<th>Exact quota (share) of house</th>
<th>Webster or Huntington-Hill number of seats</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>70,653</td>
<td>1.552</td>
<td>2</td>
</tr>
<tr>
<td>B</td>
<td>117,404</td>
<td>2.579</td>
<td>3</td>
</tr>
<tr>
<td>C</td>
<td>210,923</td>
<td>4.633</td>
<td>5</td>
</tr>
<tr>
<td>D</td>
<td>1,194,456</td>
<td>26.236</td>
<td>25</td>
</tr>
<tr>
<td>Total</td>
<td>1,593,436</td>
<td>35</td>
<td>35</td>
</tr>
</tbody>
</table>
Balinski-Young Theorem:
Michel Balinski (1933-2019)  
H.P. (Peyton) Young (now at Oxford) 

(who both taught for many years at CUNY's Graduate Center)
There is no apportionment method which obeys both population monotonicity and quota!!

(Core message: no "perfect" way to solve the apportionment problem.)
Example:

Comparing Adams, Webster (Saint-Lague), and Jefferson's (D'Hondt) Methods of Apportionment
Apportionment:

Meeting claims on a positive integer sized collection of items (size h) by giving claimants non-negative integer amounts of the items which sum to h.
Saint-Laguë (Webster's method): (Note: C eventually gets one seat, but not automatically at the start as would be true for Webster.)

<table>
<thead>
<tr>
<th>h=10</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Original claims</td>
<td>630</td>
<td>320</td>
<td>50</td>
</tr>
<tr>
<td>Divide by: 1</td>
<td>630</td>
<td>1</td>
<td>320</td>
</tr>
<tr>
<td>3</td>
<td>210</td>
<td>3</td>
<td>106.7</td>
</tr>
<tr>
<td>5</td>
<td>126</td>
<td>4</td>
<td>64</td>
</tr>
<tr>
<td>7</td>
<td>90</td>
<td>6</td>
<td>45.7</td>
</tr>
<tr>
<td>9</td>
<td>70</td>
<td>7</td>
<td>35.6</td>
</tr>
<tr>
<td>11</td>
<td>57.3</td>
<td>9</td>
<td>29.1</td>
</tr>
<tr>
<td>13</td>
<td>48.5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Webster: Each claimant gets an item at the start:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>h=10</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
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</tr>
<tr>
<td>Original claims</td>
<td>630</td>
<td>320</td>
<td>50</td>
</tr>
<tr>
<td>Divide by: 1</td>
<td>630 1-3</td>
<td>320 1-3</td>
<td>50 1-3</td>
</tr>
<tr>
<td>3</td>
<td>210 4</td>
<td>106.7 6</td>
<td>16.7</td>
</tr>
<tr>
<td>5</td>
<td>126 5</td>
<td>64 9</td>
<td>10</td>
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<tr>
<td>7</td>
<td>90 7</td>
<td>45.7</td>
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</tr>
<tr>
<td>9</td>
<td>70 8</td>
<td>35.6</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>57.3 10</td>
<td>29.1</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>48.5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
## Adams: (Divisor version round up)

<table>
<thead>
<tr>
<th>h=10</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original claims</td>
<td>630</td>
<td>320</td>
<td>50</td>
</tr>
<tr>
<td>Divide by: 0</td>
<td>∞</td>
<td>1-3</td>
<td>∞</td>
</tr>
<tr>
<td>1</td>
<td>630</td>
<td>4</td>
<td>320</td>
</tr>
<tr>
<td>2</td>
<td>315</td>
<td>6</td>
<td>160</td>
</tr>
<tr>
<td>3</td>
<td>210</td>
<td>7</td>
<td>106.7</td>
</tr>
<tr>
<td>4</td>
<td>157.5</td>
<td>9</td>
<td>80</td>
</tr>
<tr>
<td>5</td>
<td>126</td>
<td>10</td>
<td>64</td>
</tr>
<tr>
<td>6</td>
<td>105</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
D'Hondt (divisor version; round down)

<table>
<thead>
<tr>
<th>h=10</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original claims</td>
<td>630</td>
<td>320</td>
<td>50</td>
</tr>
<tr>
<td>Divide by: 1</td>
<td>630</td>
<td>1</td>
<td>320 2 50</td>
</tr>
<tr>
<td>2</td>
<td>315</td>
<td>160</td>
<td>5 25</td>
</tr>
<tr>
<td>3</td>
<td>210</td>
<td>106.7</td>
<td>8 16.7</td>
</tr>
<tr>
<td>4</td>
<td>157.5</td>
<td>80</td>
<td>12.5</td>
</tr>
<tr>
<td>5</td>
<td>126</td>
<td>64</td>
<td>10</td>
</tr>
<tr>
<td>6</td>
<td>105</td>
<td>53.3</td>
<td>8.3</td>
</tr>
<tr>
<td>7</td>
<td>90</td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Jefferson would give each claimant one item at the start.
Each divisor method optimizes one or more measure of "pairwise" fairness:

For the measure comparing claimants i and j:

$$|a_i / P_i - a_j / P_j|$$

Webster is optimal.
Surprisingly for the "reciprocal" quantities a different algorithm is best.

For the measure

\[ |P_j / a_j - P_i / a_i| \]

Dean's method is optimal.
When one uses relative rather than absolute measures of pairwise fairness then Huntington-Hill is best for all of the measures!
What fairness properties might apportionment methods obey?
Fairness axioms:

1. House size $h$ monotonocity.

If more items become unexpectedly available, one's share for the larger $h$ should not go down!
For many practical problems house monotonicity is not important and there are algorithms that obey house size monotonicity and other critical fairness properties.
2. Population monotonicity

If some state's population increases absolutely or relatively to another state, this should not result in fewer seats being given to it.
3. Quota

A state should get its fair share if it is an integer and the fair share rounded up or down to the next integer if it is not an integer.
Balinski-Young Theorem:
Michel Balinski (1933-2019)
H.P. (Peyton) Young (now at Oxford)

(who both taught for many years at CUNY's Graduate Center)
Balinski studied with Albert Tucker at Princeton - Tucker was the person who provided the "prisoner" story for "Prisoner's dilemma." (His thesis dealt with a mathematical programming problem.)
There is no apportionment method which obeys both population monotonicity and quota!!

(Core message: no "perfect" way to solve the apportionment problem.)
Bankruptcy model: 
Estate division model:

The origins of the bankruptcy model are many but include both recent and ancient work on the topic:

a. A parent has willed various amounts in his/her estate but the size of the estate $E$ is not large enough to cover all the amounts!
b. A company has gone bankrupt but there are not enough remaining assets to pay all of the persons/companies owed money.
c. Disaster relief

A fund has been established to help out those who were the victims of a flood, hurricane, earthquake, or tornado.

Size of the relief fund is not big enough to cover all the claims that are made.
Abstract version:

We have claimants (players) 1, 2,...,k whose claims $c_1, ..., c_k$ sum to more than a positive amount E. How much should be given to each claimant?

I usually number the players so that claims are increasing in size.
Comment: Some claimants may be "richer" than others and if they get a small part of their claims they will not be "hurt" badly. However, some claimants may go "bankrupt" themselves if they don't get back what they are owed!

This model ONLY looks at the size of the claims and not the "affluence" of the claimants.
The estate $E$ is usually **money** but it could be:

a. water

b. an amount of a limited medical supply

c. time (access to a time share)

d. raising taxes from different income classes (many poor; few rich)
So a typical example of such a bankruptcy problem would like like:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th></th>
<th></th>
<th>E = 120</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>60</td>
<td>90</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th></th>
<th>E = 150</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>60</td>
<td>90</td>
<td>120</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E = 180</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>60</td>
<td>90</td>
<td>120</td>
<td>120</td>
<td></td>
</tr>
</tbody>
</table>
How might one distribute the estate E to the claimants in each of these cases?
Note: Collectively the players get back all of the estate E and we can find the "loss" they collectively incur by subtracting the estate E from the total amount claimed.

Example: A owed 50, B owed 150

E = 160; so the loss incurred is 40. (Loss equals \((50 + 150) - 160 = 40\)
What fairness ideas do you use?

A  B  E = 120
40  160

There are surprisingly many appealing ways to decide how to settle the claims!
Initially I will look at examples where there are only two claimants, but realistic versions typically have many more than 2 claimants.
Common approaches:

1. Entity equity

Treat all claimants alike no matter what the size of their claims.

(Example of entity equity: US Senate - each state gets one seat regardless of its population.)
What happens when you use entity equity in this example?

A       B       E = 120
40      160

Answer: Give each player 60.
Many feel it is not "fair" to give a claimant more than they asked for!

How to modify entity equity to accommodate not giving a claimant more than he/she asked for goes back to Maimonides (1138-1204)
2. (Maimonides gain) Equalize as much as possible the amount given to each claimant but never give a claimant more than he/she asked for!

This is what is known as a constrained optimization problem. Can be solved using mathematical programming tools.
Rather than formulate these questions in symbolic terms we will use some geometric diagrams to help us reason through how to solve them in many cases.
One way to think of solving this kind of problem for two or more claimants is to think of the claims as bins and use a pitcher of fluid with the amount of fluid equal to the size of the estate to fill the bins. One can't pour more fluid into a bin than its capacity! When the pitcher runs out of fluid one is done.
Bins to help solve a claims problem with claims of 40 and 160.
This diagram illustrated how one tries to equalize payments until some claimant(s) gets all of its (their) claim(s) before the estate runs out:
E=120

(final amounts to claimants at the bottom)
3. Equalize losses between the claimants even if this means having some claimant(s) subsidize the settlement.

Does not seem fair to many people.
4. Maimonides loss

Equalize losses as much as possible without forcing any claimant to subsidize the settlement.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>E = 120</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>160</td>
<td></td>
</tr>
</tbody>
</table>

A=0; B = 120. Each loses 40.
5. Proportionality

Give each claimant the portion of $E$ that arises from its percentage of the total claims.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>160</td>
<td></td>
</tr>
</tbody>
</table>

$E = 120$

A gets 24; B gets 96.
6. Concede and divide (sometimes called the contested garment rule and sometimes the Talmudic Method).
Example:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>E = 120</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>160</td>
<td></td>
</tr>
</tbody>
</table>

B says to the person distributing the estate: A is asking for 40 of 120, so 80 of the money must be mine! This is B's contested claim against A.
In this example A has no uncontested claim against B, but it could happen both have uncontested claim.

Method: give each player their uncontested claim and slit the remainder left in the estate equally!
\[
\begin{array}{cc}
A & B \\
40 & 160 \\
\end{array}
\]

E = 120

So B gets \(80 + 20 = 100\)

and A gets \(0 + 20 = 20\)

40 units are split equally.

Final: A gets 20 and B gets 100.
6. Shapley value

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>40</td>
<td>160</td>
<td>120</td>
</tr>
</tbody>
</table>

AB - A gets 40, B gets 80
BA - B gets 120, A gets 0

Final allocation: A gets \( \frac{40+0}{2} = 20 \)
B gets \( \frac{120+80}{2} = 100 \)
7. Fixed percent on the dollar, this percentage chosen to use up the estate. (Gives same answer as proportionality.)

Our example:

\[40x + 160x = 120; \ x = .6 \] so give A=24 and B=96.
8. Proportion the loss equally.

\[
\begin{array}{c|c}
A & B \\
40 & 160 \\
\end{array}
\quad E = 120
\]

\[
\begin{array}{c|c}
A & B \\
40 & 160 \\
\end{array}
\quad \text{Loss} = 80
\]

A's share of loss is 16; B's share: 64
A's share in gain = 40-16 = 24
B's share in gain = 160-64 = 96
Note: Same as proportional gain.
Theorem:

Solving a bankruptcy problem using proportionality of gain and proportionality of loss of an estate of size $E$ gives the same values to each of the claimants.
9. Payoff the claims when $E$ grows to the amount equal to the sum of the claims in the future at current annual interest rate $i$. To pay claims now, computer present value of the claims at interest rate $i$:

Theorem: This method gives the same result as proportionality.
10. (R. Aumann and M. Maschler; Sometimes called the "Talmudic method) - This method extends concede-divide to more than 2 players. Settle half of the claims using Maimonides gain; if any of the estate is left use what is left to settle the remaining half of the claims using Maimonides loss.
How does one decide which of these methods is better or worse?
Bankruptcy Model: Fairness axioms for solution methods.

Typical example:

Claims:   $E$ is available to distribute

$A = 30 \quad B = 90 \quad C = 120 ; \quad E = 150$
Fairness axioms for a method:

1. If two claimants have equal claims they should get the same amount.

2. If more money is found to pay off the estate (E grows to a bigger E*) then what a claimant is given should not go down. (Monotonicity in estate size.)
3. Two problems differ in that one claimant Z is asking for more, the other claimants and E (estate size) are unchanged.

Z's share where her claim went up should not now be SMALLER for a fair method.

(Monotonicity in claim sizes.)
Another interesting idea:

Suppose there is a bankruptcy problem solved by method M, and M gives a subset X of claimants D dollars. Now look at the new problem where one uses the amount D to settle the original claims using D instead of E.
M is called *consistent* if the members of a subset of claimants X get the same amount when M is applied to X using D as the "new" estate size as when it used E as the estate size.

The Talmudic method is *consistent*. When applied to 2 claimants it gives the same result as concede and divide.
Other methods are consistent as well.

There are many papers concerned with exactly what axioms a method to solve a bankruptcy problem must hold in order for one to be *forced* to use a particular method. Axiomatics is important in social choice as well as geometry and algebra.
The "definitive" reference here is:

William Thomson, How to Divide When There Isn't Enough (Subtitle: From Aristotle, the Talmud, and Maimonides to the Axiomatics of Resource Allocation)

Cambridge U. Press, 2019, (481 pages)
Example: (Review)

A = 30  B = 90  C = 120  E = 150

Half claims: A = 15, B =45, C= 60

So these can be met:

A = 15, B = 45, C = 60 (120 used)
Since E was 150 we have 30 units left to settle the other half-claims:

Half claims: A = 15, B = 45, C = 60

Give C 15 units to reduce her loss to 45. Now we can give B and C each 7.5 to try to equalize losses as much as possible.
So A: $15 + 0 = 15$

B: $45 + 7.5 = 52.5$

C: $60 + 15 + 7.5 = 82.5$

These add to E = 150
A partial test of "consistency."

Check B and C. Their claims were 90 and 120. How would the these claims be settled with the 135 units they were given, using the Talmudic method? Half claims are $B = 45$ and $C = 60$; settle these with 105 units, leaving 30 left.
Now settle: B = 45 and C = 60 based on losses. Give C, 15 to reduce C's loss to B's of 45, and split the remaining 15 units between them - 7.5 to each.

B's share is $45 + 7.5 = 52.5$
C's share is $60 + 15 + 7.5 = 82.5$

So we settled B and C's claims in a consistent manner!!
(Same as concede and divide.)
We saw many ways to decide an election in an appealing manner; many ways to solve a bankruptcy problem appealingly; many ways to solve an apportionment problem in an appealing manner. However, picking which appealing approach is not so clear, and in many cases there are short lists of fairness properties so that none of these appealing methods obey ALL of the items on a short list of fairness properties! (Arrow's Theorem; Satterthwaite-Gibbard; Balinski-Young; impossibility results due to William Thomson for bankruptcy problems.)
Coalition games:
In bankruptcy problems the players (claimants) are acting on their own for a share of the "pie."

In coalitional games, single or groups of claimants want:

a. To lower their costs

b. To increase their profits

c. Improve outcomes by cooperation
Have a good week!

Questions: email me at:

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and keep an eye on:

https://york.cuny.edu/~malk/gametheory/index.html