Can the number P be achieved by playing a game with integer payoffs?

Prepared by:

Joseph Malkevitch
Mathematics Department
York College (CUNY)
Jamaica, NY 11451

demail:

malkevitch@york.cuny.edu

web page:

http://york.cuny.edu/~malk

Game 1

Consider the zero-sum matrix game (first entry is Row's payoff, and the second entry is Column's payoff.

<table>
<thead>
<tr>
<th></th>
<th>Column I</th>
<th>Column II</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 1</td>
<td>(20, -20)</td>
<td>(-5, 5)</td>
</tr>
<tr>
<td>Row 2</td>
<td>(-8, 8)</td>
<td>(2, -2)</td>
</tr>
</tbody>
</table>

Suppose Row and Column accurately report that after some unspecified number of plays Row has cumulative earnings of 19. Can one tell if this is possible? What about the case where the sum is 19 and the game was played 227 times?

If you are not sure how to solve this problem you might try a simpler case. The game above has 4 possible outcomes on any play of the game for Row, namely, 20, -5, -8, or 2. What about the game below, which perhaps seems a bit silly because Row has only one possible payoff! Note, the game is not a zero-sum game.

<table>
<thead>
<tr>
<th></th>
<th>Column I</th>
<th>Column II</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 1</td>
<td>(20, -20)</td>
<td>(20, 5)</td>
</tr>
<tr>
<td>Row 2</td>
<td>(20, 8)</td>
<td>(20, -2)</td>
</tr>
</tbody>
</table>
Question 1 (for game immediately above)

a. How would you play this game if you were Row? (Column?)
b. Can Row achieve the outcome of 19 after some series of plays?
c. Can Row achieve the outcome of 280 after some series of plays?
d. Do you see a "method" to solve questions of this type?

Now suppose we look at the game below which is a zero-sum game with only two different payoffs for Row.

<table>
<thead>
<tr>
<th></th>
<th>Column I</th>
<th>Column II</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 1</td>
<td>(2, -2)</td>
<td>(-5, 5)</td>
</tr>
<tr>
<td>Row 2</td>
<td>(-5, 5)</td>
<td>(2, -2)</td>
</tr>
</tbody>
</table>

Question 2

a. How would you play this game if you were Row? Column?
b. Can Row achieve the outcome of 19 after some series of plays?
c. Can Row achieve the outcome of 280 after some series of plays?
d. Do you see a "method" to solve questions of this type?

Now look at this game which is zero-sum and has 3 different payoffs for Row.

<table>
<thead>
<tr>
<th></th>
<th>Column I</th>
<th>Column II</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 1</td>
<td>(2, -2)</td>
<td>(-5, 5)</td>
</tr>
<tr>
<td>Row 2</td>
<td>(-5, 5)</td>
<td>(7, -7)</td>
</tr>
</tbody>
</table>

Question 3

a. How would you play this game if you were Row? Column?
b. Can Row achieve the outcome of 19 after some series of plays?

c. Can Row achieve the outcome of 280 after some series of plays?

d. Do you see a "method" to solve questions of this type?

Finally, referring to the very first game matrix above we can ask:

Question 4

a. How would you play this game if you were Row? Column?

b. Can Row achieve the outcome of 19 after some series of plays?

c. Can Row achieve the outcome of 280 after some series of plays?

d. Do you see a "method" to solve questions of this type?

It turns out that all of these questions can be approached using insights from algebra and number theory. But there is a lesson to be learned here about looking at mathematics in terms of searching for patterns, and also for different approaches to teaching problem solving. One can imagine introducing this kind of question in some kind of IBL (Inquiry Based Learning) setting. IBL does not allow students to "tune out" but even for strong students does the fact that they "get the right answers" to the questions above mean that they have understood the possible different mathematical approaches that could be used to solve problems of this kind? Solving an instance of a problem does not mean one has developed an "algorithm" for solving the problem or understands the "conceptual framework" in which the problem can be thought about.

Spoiler alert!

One approach to the problems above involves the idea of a diophantine (named to honor Diophantus) equation, an equation in which one looks for positive integer solutions for the equation. Some authors use diophantine equations to allow any integer solutions or, more restrictively, non-negative integer solutions. Linear diophantine equations in two variables (e.g. solve 5x + 11y = 27 for x and y which are integers) are an interesting extension of discussions of solving linear equations, a standard topic in American K-12 mathematics education. The topic leads one to ideas about the greatest common divisor (gcd) and least common multiple notions, and to a topic that is not always treated in K-12 mathematics education, the Euclidean algorithm for finding a gcd and ideas about congruences of integers modulo m.