Introduction

Unfortunately, natural and industrial disasters occur with distressing regularity. There are hurricanes, earthquakes, floods, tornados, industrial accidents, plane crashes, etc. These events cause individual people the loss of property, personal suffering and sometimes lives.

In the wake of a storm such as a hurricane, local or national governments (or private organizations) may set aside a dedicated "pot" of money (or clothing, tents, food, etc.) to help the people or communities affected by the storm with rebuilding or dealing with losses. Typically, the fund consists of money put aside to ameliorate the disaster. However, usually the size of the fund is not large enough to cover all of the legitimate claims that are made against the sum of money allotted to try to relieve suffering.

General modeling Situation

A natural disaster (e.g. hurricane) has caused extensive damage in a localized area. A special monetary fund of size $E$ has been created for the benefit of those who suffered losses due to the disaster. Suppose that honest claims of size $C_1, C_2, \ldots, C_r$ from claimants 1, 2, \ldots, $r$ have been received by the "administrator" charged with distributing the money from the special fund. Numbers are being used to indicate the names of the claimants, and $C_i$ is the amount claimed by claimant $i$. Unfortunately, the total amount of the claims is smaller than the amount $E$ available to pay off the claims (e.g. $C_1 + C_2 + \ldots + C_r > E$).

**Task:** Devise a fair system for the fund administrator to distribute the amount $E$ to the claimants.

Note: A variety of specific instances of this kind of problem are given in the section below. These specific instances will not only provide a test for your attempt to treat the claimants fairly but also for how to carry out your model
for specific values of the claims.

Issues for you, the modeler, to keep in mind:

a. The claimants in different settings may be a single person, a household, or a whole city.

b. Claimants may be rich or poor, or if the claimants are cities, have very different sizes in terms of wealth, economic prosperity, physical area and/or population.

c. The size of the claims in the test instances for your model are merely listed as numbers. Thus, you can’t tell from 6 whether this is 600,000 dollars or 6 million dollars, or 60 dollars. Does your view of what constitutes fairness change with an altered scale for the amount of money involved in the claims?

d. When relief from the fund is given to a claimant i, in evaluating how to value what the claimant has received back the claimant may think in terms of what was received (gain) or may view what was "lost." Does how to view gain versus loss in the way claims are settled change your way of thinking about what is fair?

e. Can you think of other situations where there is an "estate" E and the claims against E exceed the value of E? How do these other situations resemble distributing disaster recovery funds, and how are they different?

**Instances for Students**

The only information you have available is:

1. 

   Estate E = 210

   A claims 60  
   B claims 210

   Repeat for E = 40 and the same claims.

2. Estate E = 100

   A claims 40  
   B claims 40
C claims 60
D claims 160

Repeat for E = 150; E = 210; and E = 280.

3. Estate E = 300

A claims 10
B claims 20
C claims 30
D claims 440

Repeat for E = 40; E = 400.

For Teachers

Here are some ways to guide students to some of the ideas that have been used to solve problems of this kind. This particular class of problems is known as bankruptcy problems. One version of how to "set" such questions is that a firm has gone bankrupt with remaining assets E but the claims against these assets from creditors exceeds E. The reason why E is often used in bankruptcy problems is that in other settings of this problem we have an estate E and the claims against this estate are larger than the amounts of money available in the estate.

1. Entity equity. One approach might be to say each claimant should be treated as an entity, and that one should just divide the amount E equally among all of the claimants. This is the approach to giving each state in the US Senate two seats for each state no matter how large or small in population or area. Each state is treated equally.

However, this approach in some particular instances of a bankruptcy problem does allow the possibility that a claimant is given more than it claims. This will seem all right to some students but unreasonable to others. So one can modify entity equity to equalize what is given to each claimant as much as possible, but not give any claimant more than that claimant asks for. This "constrained equality of gain" approach was suggested by the philosopher Maimonides (Born 1138 in Spain; Died 1204 in Egypt) during the Middle Ages when "talmudic" scholars attempted to resolve various hypothetical bankruptcy problems.

Faced with the idea of entity equity from the viewpoint of "gain," it may be suggested that an alternate view in entity equity is dividing up the loss
equally. How much is lost collectively by the claimants? The loss $L = \text{Total claims} - E$. How the losses are "distributed" among the claimants may catch one's attention in considering if a fair result has occurred.

The analogue of the method of total equality of entity gain suggests that there be a method where each player is given an equal amount of loss. This, however, is not always possible without some players adding their own money to the estate $E$, and just as giving people more than they claim, asking some claimants to "subsidize" the settlement will be rejected by many. However, it may be possible to give scenarios where this "subsidization" might have some appeal. Just as one can have a constrained equality of gain approach as mentioned above, due to Maimonides, one can also have constrained equality of loss, without subsidization, and this system was also mentioned by Maimonides. To repeat, the idea is to equalize the loss as much as possible among the claimants until the amount $E$ is exhausted and not to enlarge the estate with contributions from the claimants themselves to achieve total equality of loss.

One appealing aspect of the method which tries to equalize the losses of the claimants is that it leads to the standard algebra topic of solving two equations in two unknowns in a natural way.

Consider this problem:

Example 1:

Suppose $E$ (remaining assets with which to settle claims) = $210

Claimant A has verified claims of $60.
Claimant B has verified claims of $300.

If we give A, $a$ units, and B, $b$ units we have the following equations:

\[
\begin{align*}
a + b &= 210 \\
60 - a &= 300 - b
\end{align*}
\]

Simplifying we get:

\[
\begin{align*}
a + b &= 210 \\
-a + b &= 240
\end{align*}
\]

Hence, $2b = 450$ and $b = 225$, which means that $a = -15$. 
Whoops? What does the -15 mean? The only way to equalize the loss total is for A to add 15 dollars to the estate, so A's loss will be $60 + 15 = 75$, and B's loss will be $300 - 225 = 75$ also.

If we are not happy with this approach we can use Maimonides loss, which will mean that we try to make the losses of A and B as equal as possible. Thus, give B 210 of the 210 units of E, thereby making at this stage A having a loss of 60 units and B a 90 unit loss. Without A "subsidizing" the value of E with more units, there is no way to make the losses of both more equal, so the Maimonides loss solution gives A 0 units, and B 210 units.

It might seem as if to solve the constrained optimization problems involved in Maimonides gain and loss, one would have to work with equations or inequalities as we did in trying to equalize losses as indicated above. However, there is an elegant "visual" method which vastly simplifies thinking about these two methods of attacking fairness algorithms for bankruptcy problems.

Think of the estate as a colored homogeneous fluid (blue in our diagrams) and the claims as "glass containers" whose heights are the claims. We imagine that we have a system of filling up the containers (bins) that represent the claims with the fluid, with the filling stopping when one reaches the top of a claims container.
Figure 1 (Small equal amounts have been distributed to the two claimants.)

The order in which one lists the containers is somewhat a matter of discretion. Here (Figure 1) we have filled the claims containers equally with a small amount of fluid. We continue to fill the two containers equally until the smallest claim is "topped" off when it reaches the height of the claims container.
At this juncture (Figure 2) we have filled both containers to the height of 60. Since there are two containers at this height we have used 120 units of the estate thought of as fluid. So we have 210-120 or 90 units of fluid left. This (Figure 3) allows us to continue adding to the container asking a claim of 300, to the height of 60 + 90 or 150. We have used 150 + 60 or 210 units - the total value of E. Note the diagrams above are "symbolic" and don’t have to be metrically accurate in order to do the arithmetic that solves the problem.
The same containers can be used in one's mind's eye to solve constrained loss equality problems. Defying the laws of gravity one needs to pare down the loss of largest claimant, if possible to the loss of the next largest claimant so as to try to equalize their losses. In this example if we start to "fill" the 300 container starting at its top, we can put 210 units of fluid, all of the estate before reaching the level at the top of the other claimant, whose container has height 60. Thus, to try to equalize the claims, all of the estate E of 210 must be given to claimant B, with 0 given to claimant A.

Another approach often "invented" by students is:

Assign each claimant an amount proportional to the size of its claim.
In fact, the idea of this approach is sometimes triggered by the fact that when discussing how the US Senate treats all states equally, students will raise the idea that in the US House of Representatives states be given a number of seats which is in proportion to their population. Since states can not be assigned a fraction of a seat, the equity problem in designing a fair way to give states seats in the House of Representatives has rather different ramifications than the bankruptcy problem and is known as the apportionment problem. The apportionment problem for the US House of Representatives (there are also other interesting apportionment problems, for example, assigning parties seats in European parliaments in proportion to the vote they get in elections) has the additional Constitutional requirement that each state get at least one seat. For bankruptcy problems, there are methods where a claimant will be given no part of E at all!

In light of the discussion above one could also assign each claimant an amount computed from first distributing the total loss proportionately based on the claims, and then giving the claimants what they deserve from this point of view. Using a bit of algebra it is possible to see that the two seemingly different approaches in fact give each claimant the same amount from E.

There are two other approaches to the proportional way of assigning claimants parts of E. One of these is based on the following idea. Since E is not enough to pay off the claimants, why not invest E until it grows to the total claimed at the current rate of interest and then disperse the amounts necessary to pay off the claimants? However, this would entail having the claimants perhaps having to wait a long time for their money. So, after computing the amount of time that it takes E to grow to the total claim, compute the present value of those future claims. It is not hard to see that these present values would assign each claimant the proportional solution.

Another way to view the proportional solution would be to say, since we can’t pay off the claimants fully, how about give each claimant c units for each each unit of claim they have, where c is less than one unit. (This can be thought of as giving each claimant c cents for each dollar of a monetary claim.) It is not hard to see, using some algebra, that this approach, too, yields the proportional solution.

Note that when proportionality is used, no claimant is ever given all of their claim. However, with some of the other fairness methods above some claimants will get all of their claims but others won't. This bothers some people, but others feel that this is what fairness requires.
Summarizing, we have now looked at 6 different ways of thinking of solving bankruptcy problems:

a. Entity equity for gain
b. Entity equity for loss
c. Maimonides gain (constrained equality of gain)
d. Maimonides loss (constrained equality of loss)
e. Proportionality of gain
f. Proportionality of loss

But there are other ideas!

Suppose that one picks an ordering of the claimants at random, those not giving any advantage to any claimant, and disperses the amount $E$ as follows.

i. Starting with the claimant at the top of the ordering give that claimant all of what is claimed or if the claim exceeds the amount $E$, just all of $E$.

If there is more money left, then

ii. Repeat i. with the second person on the ordering.

This approach is repeated until all of $E$ is dispersed.

However, while it may seem "evenhanded" to use a random approach, many find this approach unappealing. But, there is a germ of an idea here that does restore fairness in some people's minds. Consider all orders of dispersing the estate $E$ to the claimants, as above, that is, give the money out in the order the claimants appear in a given one of the $(r!)$ orderings, and then take the mean of the amounts that each claimant gets over all of the possible $r!$ orderings ($r$ claimants, $r!$ orderings).

This solution, which appears in various guises in different parts of game theory, is known as the Shapley Value, and it is part of the reason why Lloyd Shapely won a Nobel Prize in Economics in 2012 (shared with the game theorist Alvin Roth).

Like the many appealing and different methods that can be used for deciding
an election using ranked ballots, we have seen now a variety of appealing methods for solving bankruptcy problems. When having students model bankruptcy problems, how many claimants is it wise to have in sample instances that students might consider? While realism requires problems with reasonably large number of claimants, and certain aspects of the way the different methods distribute the estate E seem more reasonable when one has many claimants, there is the modeling education idea that one should try simpler cases than the actual problem at hand in order to get insight. For bankruptcy problems there is another reason to look at the two-claimant case that seems very appealing and which has no natural counterpart for bankruptcy problems with more than two claimants.

I will describe a method for solving a two-claimant problem which appears in discussions of bankruptcy problems from the Middle Ages, and which, I, at least would never have thought of in a million years!

Let me illustrate with this example, and here as in the instances above, I will use letters A, B, etc. for the claimants' names and the size of their claims, and the letters a, b, etc. for the amounts that a given method will give to the claimants. As usual, E will stand for the size of the E estate, so I will not use examples with more than 4 claimants. Also, I will for convenience list the claims in non-decreasing order of size.

E = 200

A's claim: 120; B's claim 180

Here is how some scholars attacked this problem in the Middle Ages:

Suppose A goes to the person administering the 200 units of an estate and says: Look, there are only two claimants, me A and B. B is only claiming 180 so 20 units of the estate must be mine. This amount, 20, is called A's uncontested claim against B. For some values of the claims of A and B, B cannot make a similar argument (because A is asking for the whole estate or more) but in this example, B has a similar argument. B can go to the administrator dispersing the estate E and say, A is only asking for 120 so 80 units of the estate must be mine. So B's uncontested claim against A is given by 80 units. Now each of A and B are given their uncontested claims, which add to 100 units. This means 100 units still to be distributed (200-100). Since the remaining amount (100) is contested by both A and B the administrator splits this equally between A and B. So A gets 20 + 100/2 = 70 and B gets 80 + 100/2 = 130. As a check, 70 + 130 equals the size E of the estate, 200. Note that it is not totally obvious that there will always be additional funds to
distribute after the uncontested claims are met, but there always will be.

This procedure is sometimes called the Talmudic Method, sometimes the contested garment rule, and sometimes concede-and-divide. The use of the term contested garment rule to describe this method stems from problems that are set as two claimants contesting a single garment. But bankruptcy problems are usually set in situations where subdivision of the estate $E$ involves distributing something that is homogeneous and "infinitely divisible." Strictly speaking money is not infinitely divisible because there is a smallest unit of currency, in America, a penny. However, the value of a garment is destroyed when it is subdivided to give the parts to claimants. Bankruptcy problems are usually stated in terms of distributing something whose value is not destroyed when it is subdivided, as would be the case for a garment, painting, or other kinds of assets, which might be in an estate. Thus, in recent bankruptcy literature the method called "the contested garment rule" is now often called concede-and-divide.

The recent interest in looking at bankruptcy problems was stimulated by a paper by the political scientist Barry O’Neil. The problem stimulated Michael Maschler and Robert Aumann to do some seminal work on the problem. O’Neil called attention to a problem described in the Babylonian Talmud summarized in the table below, which shows solutions for three different bankruptcy problems with three claimants.

<table>
<thead>
<tr>
<th>Estate</th>
<th>100</th>
<th>200</th>
<th>300</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>33 1/3</td>
<td>33 1/3</td>
<td>33 1/3</td>
</tr>
<tr>
<td>200</td>
<td>50</td>
<td>75</td>
<td>75</td>
</tr>
<tr>
<td>300</td>
<td>50</td>
<td>100</td>
<td>150</td>
</tr>
</tbody>
</table>

Table 1 (Three bankruptcy problems described in the Babylonian Talmud.)

The entries in the body of the table represent solutions which correspond to the estate sizes of 100, 200, and 300. The top line looks like equal division while the bottom line looks like proportionality, but the middle row of numbers is puzzling. Aumann and Maschler showed there was one method, now usually called the Talmudic Method, which will explain all three rows!

What Aumann and Maschler did constitutes a "mathematical detective story," which is interesting in its own right and can be found in Malkevitch [2005] but here I will just describe the ingenious algorithm that they found for explaining
the numbers in Table 1. The method can be applied to any of the rows of the table, but I will show how it applies to the puzzling second row of the table.

The basic idea is to divide the claims of all the claimants in half, to get "half-claims" for all of the claimants. First, take the first collection of half-claims and use the estate E to settle the claims using the Maimonides gain method. If there is money left over, use it to settle the other collection of half-claims using the Maimonides loss method.

So for an estate of size 200 (Table 2, second line), A, B, and C's "half-claims" are 50, 100, and 150.

Applying Maimonides gain, we try to give A, B, C as equal amounts as possible but not exceeding their claims. So we give A, B, and C 50 each, at which point we have distributed 150 of the 200 units available. Now we continue to treat B and C equally, giving them each 25 additional units, which gives A 50 units, B 75 units, and C 75 units, which exhausts the estate E and so we are done. (Typically, however, one might have enough of value in the estate that after the first half-claims are met using Maimonides gain, one needs to work on the second half-claims using Maimonides loss.)

The method of Aumann and Maschler (let me call it the Talmudic Method) when applied to the case of 3 or more claimants satisfies a remarkable property known as consistency. Suppose one looks at a subset T of at least two claimants but not the full collection of claimants. Suppose that the Talmudic Method assigns T an amount E*. Then the Talmudic Method, when used to settle the same sized claims but using E* instead of E, distributes the amount E* to the members of T with exactly the same amounts as in the original solution.

In particular, if T has exactly 2 members, then the amounts distributed are what would be given by the concede-and-divide (contested garment) method.

For an illustration of the idea of consistency, consider the Talmudic Solution for an estate of 200, the second row of Table 2. The amount assigned in this case is A = 50, B = 75, C = 75.

To check "consistency" in this example requires that one check three cases:

a. E* = 125, for claims of A = 100 and B = 200

b. E* = 125 for claims of A = 100 and C = 300
c. $E^* = 150$ for claims of $B = 200$ and $C = 300$

For a., A has no uncontested claim against B but B has a 25 unit uncontested claim again A. So give B 25 units and split the remaining contested claim of 100 equally. A gets 50 and B gets 75, the same amounts as originally.

For b. A has no uncontested claim against C but C has a 25 unit uncontested claim again A. So give C 25 units and split the remaining 100 equally. A gets 50 and C gets 75, the same amounts as originally.

For c. Neither B nor C has an uncontested claim against the other, so the estate of 150 is equally divided. B gets 75 and C gets 75, as was true for the original settlement.

Consistency seems like a strong condition and it is remarkable that the Talmudic Method, with its balance of attention between gains and losses, satisfies this condition.

Many "fairness axioms" have been proposed that appealing methods to solve bankruptcy problems should obey. For example, if the estate goes from E to E' which is larger, and the claims stay the same, then no claimant should get a smaller amount when the method is applied using E' rather than E. If two bankruptcy problems differ only in that one claimant's claim has gone up, then a reasonable method will not give that claimant less than when the claim was smaller. Researchers on bankruptcy problems have worked towards determining which methods obey which fairness rules, and ideally towards being able to say that if one wants a particular collection of fairness rules to hold, then the only method one can use is some specific method.

In recent years much of the effort in understanding the bankruptcy problem has been carried on by William Thomson, a mathematical economist who teaches in the Economics Department at the University of Rochester. Thomson's extensive research (and that of his students) has greatly elucidated the remarkably rich collection of methods that solve bankruptcy problems and the fairness axioms these methods obey. Thomson's recent survey (Thomson [2015]) is a treasure trove of fascinating results and ideas.

Bankruptcy problems are but one of a wide range of fairness issues that can be modeled by K-12 students. Others include elections, apportionment, cost allocation, and weighted voting. Remarkably, like the modeling done above, while the theory may involve work well beyond what can be treated in K-12, often the algorithms involve little more than arithmetic and algebra!
References


Malkevitch, J., 2005 Resolving bankruptcy claims, American Mathematical Society, (web column)


Thomson, W., Consistent solutions to the fair division problem when preferences are single-peaked, J. of Economic Theory, 63 (1994) 219-245.


