

# Game Theory Sampler (2019)

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# Games I

Players named Row and Column independently choose a row (choice of two) and column (choice of two), respectively. The "payoff" is shown in the cell at the intersection of the chosen row/column pair. For example if Row plays Row 2 and Column plays Column I, Row loses 8 and Column wins 3.

	Column I	Column II
Row 1	(-4, -4)	(3, -8)
Row 2	(-8, 3)	(1, 1)

i. How should Row and Column play this game to achieve his/her best result if the game is played:

a. Once?

b. Many times?

ii. Can you think of "real world situations" where this game would be a suitable model?

## Games II

Players named Row and Column independently choose a row (choice of two) and column (choice of two), respectively. The "payoff" is shown in the cell at the intersection of the chosen row/column pair. The payoffs shown are from Row's point of view. Column's payoff is the negative of the number shown.

	C <sub>1</sub>	C <sub>2</sub>
R <sub>1</sub>	5	-2
R <sub>2</sub>	-20	8

- i. Is this a fair game?
- ii. How should Row and Column play the game to achieve his/her best result if the game is played:
  - a. Once?
  - b. Many times?

# Two-sided Markets

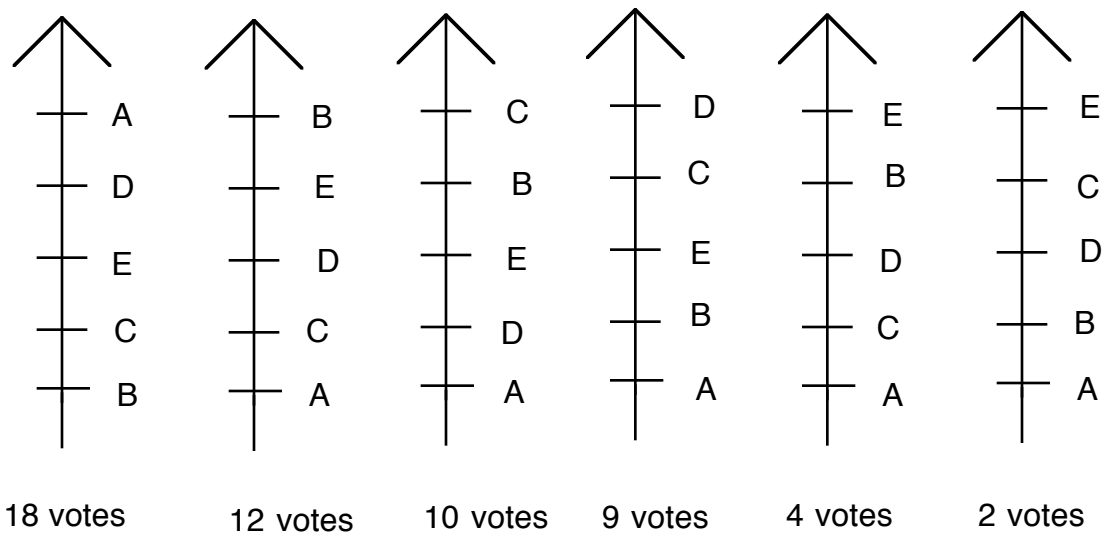
8 hospitals (left) and 8 medical students (right) have ranked each other as indicated in the tables below. For example Hospital 5 has ranked Medical Student 2, second, While Medical Student 4 has ranked Hospital 2 fourth.

1	5	7	1	2	6	8	4	3	1	5	3	7	6	1	2	8	4
2	2	3	7	5	4	1	8	6	2	8	6	3	5	7	2	1	4
3	8	5	1	4	6	2	3	7	3	1	5	6	2	4	8	7	3
4	3	2	7	4	1	6	8	5	4	8	7	3	2	4	1	5	6
5	7	2	5	1	3	6	8	4	5	6	4	7	3	8	1	2	5
6	1	8	7	5	5	4	2	3	6	2	8	5	3	4	6	7	1
7	2	5	7	6	3	4	8	1	7	7	5	2	1	8	6	4	3
8	3	8	4	5	7	2	6	1	8	7	4	1	5	2	3	6	8

- i. What is a reasonable way to pair off the hospitals and medical students?
- ii. Can you give other "real world" situations that have a similar "flavor" to this problem?

# Elections

The ballots below show the rankings of 55 faculty for 5 choices of textbooks. Which book deserves to be selected as the choice for the group?



(Note: There is no indifference; higher preferences towards the top.)

# Bankruptcy

There are three merchants who have verified claims against the remaining assets of a small company. What would be a fair way to settle the claims in each case?

## Example 1:

Claimant	A	B	C	(Assets Remaining: \$200)
Claim	\$30	\$100	\$100	

## Example 2:

Claimant	A	B	C	(Assets Remaining: \$200)
Claim	\$30	\$200	\$300	

## Example 3:

Claimant	A	B	C	(Assets Remaining: \$200)
Claim	\$50	\$100	\$150	

# Apportionment

The four divisions of a college, Arts, Sciences, Business, and Teacher Education have enrollments of:

2,180

1,880

1,420

640

A wealthy alumna of the college has decided to donate 30 scholarships to attract new students based on the different divisions' enrollment.

What is a fair way to assign the 30 scholarships to each division?

What would be a fair assignment if there were 31 scholarships?

*This problem involves the same ideas that will be needed to assign 50 states 435 seats in the US House of Representatives after the Census in 2020.*

# Fair Division

Bob (B) and Alvin (A) are siblings and have each, without the input of the other, assigned a dollar value to several items that their mother Nancy has willed to them. Each of the brothers has enough cash to make payments to the other in order to equalize any possible division that is suggested.

	A	B
Turkish Rug	\$1100	\$1800
Stamp Collection	\$1900	\$1200
Book Collection	\$20000	\$30000
Ethnic Jewelry	\$6000	\$5000

What would be a fair way to handle the situation? (What would you do if the brothers had listed the same amount for some items?)



# Cost Sharing I

Towns A and B with populations of 12 and 8 (in hundreds of thousands) need to build a new sewage treatment plant.

Stand alone costs (in millions of dollars):

$$\{A\} = 160 \quad \{B\} = 90$$

$$\text{Joint cost: } \{A, B\} = 200$$

How should A and B proceed?

# Cost Sharing II

Towns A, B, and C with populations of 12, 10, and 8 (in hundreds of thousands) need to build a new sewage treatment plant.

Example 1:

Stand alone costs (in millions of dollars):

$$\{A\} = 150 \quad \{B\} = 120 \quad \{C\} = 90$$

Joint costs:

$$\{A, B\} = 240 \quad \{A, C\} = 220 \quad \{B, C\} = 180$$

$$\{A, B, C\} = 325$$

How should A, B, and C proceed?

Example 2:

Stand alone costs (in millions of dollars):

$$\{A\} = 150 \quad \{B\} = 120 \quad \{C\} = 90$$

Joint costs:

$$\{A, B\} = 240 \quad \{A, C\} = 220 \quad \{B, C\} = 180$$

$$\{A, B, C\} = 315$$

How should A, B, and C proceed?

# Legislative Fairness

The five towns in Rural County have populations of 7, 4, 3, 3, and 1 (in hundreds of thousands), respectively.

Is it fair for the representatives of these 5 towns to cast 7, 4, 3, 3, and 1 vote, respectively, where 10 votes are required to take action in the county legislature?

# Weighted Voting Games

[12; 7, 6, 5, 2, 1] is a notation for an election system known as weighted voting. In this example there are 5 players, 1, 2, 3, 4, and 5 who cast a block of votes of size 7, 6, 5, 2, and 1 respectively. A group of players (coalition) can take "action" if the sum of the weights they cast exceeds the quota  $Q$ , in this case 12. Thus, in this example players 2, 3, and 5 can pass a bill. {2, 3, 5} is a winning coalition.

List all the winning coalitions for this "game." How might one measure the "power" of the different players in this voting environment?

How do you think the value of  $Q$  was arrived at? Does this value for  $Q$  seem reasonable?

# Weighted Voting Games II

Look into the role of the Electoral College in electing the President of the United States.

Is the Electoral College an example of a weighted voting game?

What is the power of the players in this game?

Can a candidate win the popular vote but lose in the electoral college?

# Transplants

Which of these factors should be taken into account in deciding whether or not someone is a suitable candidate for a transplant?

How famous the patient is?

How wealthy the patient is?

How old the patient is?

How sick the patient is?

How long the patient has been waiting for a transplant?.

# Measuring inequality

i. What is the difference between wealth and income?

ii. How might one measure the extent to which there is income or wealth inequality in two different countries?

iii. Should markets set the pay that a worker receives or should there be regulations which require "equal pay for equal work?"

# School Choice

Students rank schools they want to attend and schools by law are allowed to have "priorities" for students they accept. Here is information about student preferences and the priorities the schools have for the students. What assignment of students to schools would be in the interests of "society?" What assignment is best from the student's point of view? from the school's point of view?



Example: Student preferences for public high schools, where each school can only accept 1 student of this "collection" of three. For example Student 2 ranks HS1 highest (first) and Student 3 ranks HS3 third (last)

	HS1	HS2	HS3
S1	2	1	3
S2	1	2	3
S3	1	2	3

School priorities for students: (same style as above in interpreting the array.)

	S1	S2	S3
HS1	1	3	2
HS2	2	1	3
HS3	2	1	3

Here we had three schools and three students, with "capacity" one for each school. How might your "algorithm" work if schools had different capacities? If the total capacity is more than the number of students?

## Short list of important references:

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