Basic Ideas in Apportionment (2019)

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Suppose we have three "states" A, B, and C with populations 50, 80, and 70 to which we want to assign a non-negative integer number of seats in a "parliament" which has size h (for house size) where h equals 11 seats. How many seats should be given to each state? The question is similar to how many delegates at, say, the Republican Convention, for State X should be assigned to each candidate based on the vote he/she got in the Republican Primary held in State X. Fairness would seem to require some kind of "proportionality" in how many of a legislature's seats each geographical/political unit gets. For the United States House of Representatives, if a state has 30 percent of the population, presumably it should get 30 percent of seats.

It turns out there are many different "fairness" approaches to solving this kind of problem, which is known as the *apportionment problem*. While many of these methods will give the same answer, they approach the problem from very different points of view, and if one is "clever" one can construct examples, albeit often artificial ones, where the different methods give different answers. This parallels our experience with election methods where very different reasonable methods yielded different winners.

One natural way to attack this problem is to compute the total population of all the states, and then find the fraction of this population which each state has. In this example, the total population is 200 people. Hence:

A has 50/200 = .25 or 25 percent of the total population; B has 80/200 = .40 or 40 percent of the total population; C has 70/200 = .35 or 35 percent of the total population.

If we try to assign seats proportional to population and there are 11 seats, then A would be entitled to .25(11) = 2.75 seats, B would be entitled to .40(11) or 4.40 seats, and C would be entitled to .35(11) = 3.85 seats. These numbers add to 11 but the trouble is that we can't have fractions of a person as a representative. So what do we do?

Before trying to answer that question let us consider another way to think of these numbers 2,75, 4.40, and 3.85 which are sometimes called the states' exact quota, ideal quota or fair share. Since we have 200 people and 11 seats, ideally we should have 1 representative for every 200/11 people, or one representative for 18.181818..... people. This number, 18.18.18... is called the ideal district size. If we divide 50, 80, and 70 by the ideal district size, we again get the numbers, 2.75, 4.40, and 3.85 as the laws of arithmetic require.

While in some rare cases the fair share numbers we get might exactly be integers, this is not typical of what will happen. So usually we will have to give a state MORE or LESS than its fair share. It isn't possible to do "better," so what should we do? One principle would be to say that if a state X is entitled to u.xmy seats (u.xmy is an integer, plus a fractional (decimal) part - for example, 4.537 would involve the integer 4 and the fractional part 537/1000), then X would get u seats when the fractional part is zero, and otherwise u seats (u.xmy rounded down) or u + 1 seats (u.xmy rounded up). The functions that take any real number and round it down or up to the nearest integer are called the floor function and ceiling function, respectively. Clearly, even when two states don't have the same exact fair share, they may have to be assigned the same number of seats because to assign integer values there is no way around this. Thus, states may be over-represented or under-represented. Again, there is no way around this. However, one tries to use some fairness principle decided on in advance to solve the problem so that one cannot be accused of "favoritism." One principle of fairness for a method of apportionment (an algorithm which for any populations and house size which assigns non-negative integers to the states as their shares) is that it obey QUOTA. Quota means that the state gets either its fair share when it is an integer or the closest integer above or below its fair share (thus, the ceiling or floor function applied to fair share share). The apportionment method the United States currently uses to apportion the House of Representative does NOT obey quota. This method is called the Huntington-Hill method.

So how many seats should we give to A, B, and C in this problem? Perhaps the simplest method to describe is the one suggested by Alexander Hamilton, America's first Secretary of the Treasury. Hamilton suggested the following system. Assign each state the integer part of its fair share. So if a state's exact quota is u.xmy, give it u seats. If there are any seats left over, list the fractional parts in decreasing order and give away the remaining, left over seats, in order of the size of these fractional parts, states with larger fractional parts getting seats first. This method is now known as Hamilton or largest remainders. It was used during the early history of the United States over a method known today as Jefferson's method.

For our example, A would get initially assigned two (2) seats, B four (4) seats, and C three (3) seats. The fractions in order are: .85 (for C), .75 (for A), and .40 (for B). Since there are two extra sets to be assigned (11 - (2 + 4 + 3) = 2), these are given to C and A, with the result that A gets 3 seats, B gets 4 seats, and C gets 4 seats. Unfair? Yes, but there is no way to give each state its exact fair share. A is over represented, B is under represented and C is over represented.

Let us try a different system to see if that might seem more "satisfying."

Each state has an exact quota. What happens if we round these exact quotas to the nearest integer, using the usual rounding rule (which states that if the fractional part is 1/2 (e.g. .5) or above, round up to the next closest integer, otherwise one rounds down.

With this system, A's fair share is rounded from 2.75 to 3, B's fair share is rounded from 4.40 to 4 and C's fair share is rounded from 3.85 to 4. Since 3 + 4 + 4 = 11 we have assigned 11 seats and we are "happy." This solution, by coincidence, gives the same answer as Hamilton's Method. However, what would one do if instead of assigning 11 seats, more or fewer seats were assigned? What one does is to take the ideal quota (in this case 18.1818...) and adjust it up or down to obtain an AQ, adjusted quota, so that when the populations of the states are divided by AQ and rounded by the usual rounding rule, exactly h seats are assigned. Usually there is a range of AQ values that work and finding an AQ can be done by moving the exact Q quota up or down a little bit at a time so that one gets the desired result. That there must be some AQ that works is guaranteed by a continuity argument. (Caveat: in some rare cases there may be no way to assign exactly h seats due to issues involving ties. In that case one has to have a tie breaking scheme that breaks ties. For example if the house size is 15, and there are 17 states of equal size 122, some tie breaking scheme is needed.) This system (rounding using the usual rounding rule and if necessary to distribute h seats one uses an AQ instead of the ideal quota) is known in the US as Webster's Method, and in Europe as St. Lägue's Method. Note, however, that Webster's Method as I will use the term requires that each state be given one seat even though the divisor method procedure doesn't guarantee that this happens automatically since the Constitution requires this for the US House of

Representative apportionment. St. Lägue's method does not require each "state" to get one seat, though in other regards it is identical to Webster. Again, Webster's method unlike other apportionment methods (such as Adams and Huntington-Hill) does not automatically give each state one seat.

Let us try to use Webster's Method with a different house size and the same populations. Suppose h = 14. Now .25, .4, and .35 of h, gives 3.5, 5.6, and 4.9 seats, respectively. Note that these numbers add to 14 if we have done our arithmetic correctly. So when we round, we get 4, 6, 5 which add to 15, not 14. The problem was that dividing by the exact quota 14.285714... (200/14) gave away too many seats. So we adjust 14.285714... higher to make the values of the populations over the adjusted quota smaller, so that when we round we assign only 14 seats. Thus, suppose we adjust 14.285714... up to 14.3. When we divide the populations of the 3 states by 14.3 we get: 3.497 (rounded to 3 decimal places), 5.594, and 4.895. So rounding in the usual way we get 3, 6, 4, respectively, which adds up to 14 so we get a "legal" apportionment of h = 14 seats. (Had the adjusted quota we tried NOT worked we would have modified the adjusted quota until we got a value that did work) For h = 14 all the fractional parts turned out to be bigger than 1/2 so we gave away too many seats using Webster. The opposite situation is illustrated when h = 21. Computing .25(21), .4(21), and .35(21) we get 5.25, 8.4, and 7.35, respectively. Rounding gives 5, 8, 7 seats which only distributes 20 of the desired 21 seats. So, instead of using the ideal quota: 9.523809... we use the ideal quota adjusted down, say, to 9.5. Dividing 9.5 into 50, 70, and 80 gives: 5.263, 8.421, and 7.368. The fractional parts have not gone up enough so we have to go lower than 9.5 with our adjusted quota. What about using 9? Now all the quotients have fractional parts above .5, so we give away too many seats! What about 9.3? Two of the fractional parts are more than .5 so we still give away too many seats. If we use 9.4 we get as the quotients of 50, 80, and 70, the values 5.376, 8.511, and 7.447. Rounding we get 5, 9, and 7 seats which adds up to the required 21.

The trial and error way we looked for an adjusted quota can be improved upon somewhat by locating the state where "tipping" with respect to .5 will occur but it turns out, due to the extraordinary work of the mathematician E.V. Huntington (Harvard), there is a conceptually much easier way to apply Webster's apportionment method for relatively small values of h that we will learn about.

Now for an inspired idea. Why is the standard rounding rule used? Couldn't we use some other form of rounding to get other apportionments? The answer is, yes we can. Suppose we use the "spirit" of this method (round as below based on the fair share, but if h seats are not given away use an adjusted quota) using:

a. Always round down from the fair shares - we get Jefferson's Method.

b. Always round up from the fair shares - we get Adam's Method.

c. Round using the geometric mean - we get Huntington-Hill's Method.

d. Round using the harmonic mean - we get Dean's Method.

These names as used above will connote that it is required to give each state one seat to begin with. The European version of these methods will not have this requirement and, when they were used, have different names. For example the European version of Jefferson was developed by the Belgian lawyer Victor D"Hondt.

To give you the flavor of what is involved let us return to the original 11 seats and the values for the quotas of 2.75, 4.40 and 3.85. Rounding down gives 2, 4, and 3 seats for a total of 9 - two short. So instead of using the ideal quota of 18.1818.... we will use 16.5. Dividing 50, 80, and 70 by 16.5 we get: 3.030, 4.242, and 4.84. Rounding down we get, 3, 4, and 4 which now add to 11. So we give A 3 seats, B 4 seats, and C 4 seats.

You may feel that this is a lot of "words" for three methods that give the same answers. Yes, in this case all three methods yield the same answers but this is not always the case.

Apportionment questions are not limited to the assigning of seats in a parliament. Whenever one has an integer number of objects that can't be subdivided but must be assigned whole on the basis of "data" about the different claimants one can use the apportionment model.

Examples:

1. A college has been granted 7 new lines by the state legislature. What is a fair way to assign these lines to three "schools" (business, arts and sciences, education) based on the number of graduates these departments generated on average for the last 5 years?

2. How should 30 new computer systems be assigned to 6 divisions of a small company based on the sales of the previous year?

Reference:

M. Balinski and H. Young, Fair Representation, 2nd edition), Brookings

Institution, Washington, 2001.

Note this book is a "classic" and is divided into two parts. The first part is expository writing about the apportionment question in the US and its history and the second part gives some of the details of a mathematical treatment.