Basic Ideas in Apportionment-2 (2019)

Prepared by:

Joseph Malkevitch
Mathematics Department
York College (CUNY)
Jamaica, NY 11451

email:

malkevitch@york.cuny.edu

web page:

http://york.cuny.edu

Suppose we have three "states" A, B, and C with populations 50, 80, and 70 to which we want to assign a non-negative integer number of seats in a "parliament" which has size h (for house size) where h equals 11 seats. How many seats should be given to each state?

Historically, there have been 6 methods to solve this problem, known in America as: Hamilton, Adams, Huntington-Hill, Webster, Dean, and Jefferson. Among the divisor methods, Dean and Huntington-Hill are bit more complex because they involve the Harmonic Mean and the Geometric Mean. For somewhat simpler calculations we could use Webster's method, we could use Jefferson's method, or we could use Adams's method. For a particular problem such as the one above, these methods might yield the same or different numbers of seats to the different "claimants." In this particular case they lead to the same numbers of seats for h=11 but can differ for other house sizes.

We saw that using the divisor style algorithms for these methods involved:

a. Computing the fair share (ideal number of seats) for each state. (This is done by dividing the ideal district size into the population of each state. Comment: the result is rarely an integer but can be an integer.)

b. Obtaining an integer number of seats for each state by using an appropriate rounding rule. (Comment: Round up for Adams, round down for Jefferson, use the usual rounding for Webster.)

c. If exactly h seats are assigned in b. we are done. Otherwise we have to compute a new "adjusted fair share" by adjusting the ideal district size, so that after the rounding of the particular method is done, we give away exactly h seats.

From this point of view, all the h seats are assigned at once, based on the way the algorithm works. However, using the remarkable work of the mathematician E. V. Huntington, (while he was a professor of mechanics at Harvard) there is an altogether different point of view that leads to the same answers. Huntington showed that for the 5 classical divisor apportionment methods (Adams, Dean, Huntington-Hill, Webster, Jefferson), there is in many ways a much simpler algorithm to implement, and for small values of h, a lot easier for middle school and high school students to use to "get answers."

The idea is that instead of "giving out" all of the h seats at once, one gives the fixed number of h seats away one at at time until one has given away h seats. At each stage, the "state" that gets the next seat is most "deserving" from some equity point of view. This method, which I call the "table method" but is more usually known as the rank index approach, is discussed below for the Adams, Jefferson, and Webster methods.

Just as the difference in viewing divisor methods only relates to what rounding rule is used, the table method differs in what choice of numbers is used to divide the original populations to construct the tables. Here is the initial table one starts with, showing the populations of the states in decreasing order in the top row, from left to right:

	B	C	A
Pop:	80	70	50
Row 1			
2			
3			
4			
5			

The number of rows of the table will vary depending on the number of seats h that have to be distributed.

General procedure: (applies to all methods)

Once the table is prepared, the seats are given out one at a time and the state that gets the next seat is the state having the largest entry in its column that has not yet been used to assign a seat. If several states have the same entry, then one has a "tie." If the house size one cares about is bigger than where the group of ties occurs this is not a problem. If the house size one needs to assign occurs within a tied group, one needs to use some randomization device to break the tie.

Here is the scheme that is used:

0th Row (initialize), the populations listed in decreasing order from left to right.

Now the entries in the 0th row are divided by a number to get the entries in the following rows:

In the formula's below, a denotes the number of seats assigned:

Adams:

Formula: a

Adams: divide by 0, 1, 2, 3, 4, 5, 6,....

Comment: We will think of dividing by 0, which we know can't be done, as giving the outcome, infinity. Thus, Row 1 will consist of the entry infinity in every column. This in essence "forces" us to give one seat to every state as discussed below.

Jefferson:

Formula: a +1

Jefferson: divide by 1, 2, 3, 4, 5, 6, ...

Formula: a + 1/2

Webster:

Webster: divide by 1/2, 3/2, 5/2, 7/2, ...

However, because seats are assigned by size order in the table, if we multiply all table entries by 1/2, we can use the following simpler numbers:

Webster: divide by 1, 3, 5, 7, 9, ...,

e.g. First row divide by 1, second row divide by 3, third row divide by 5, etc.

Dean:

Formula: Harmonic mean of a and (a+1)

Exercise: What are the numbers used to divide the rows by?

Huntington-Hill: (In older literature sometimes called the Method of Equal Proportions)

Formula:

Square root((a(a+1))

Huntington-Hill: Divide by: 0, square root of 2, square root of 6, square root of 12, ...

Note: This is the actual method that is used to apportion the US House of Representatives at the current time. Note that whereas accidental ties can occur in a table for the other methods (with "nice integer" numbers), because we are working with rational numbers and integers, for Huntington-Hill " it seems that ties cannot occur because the numbers involved are irrational.

So now let us look at how our particular example works in the "table method" context. Here is the table prepared to distribute seats using Jefferson's method for the example we have been studying.

	B	C	A
Pop:	80	70	50
Row 1	80	70	50
2	40	35	25
3	26.67	23.33	16.67
4	20	17. 5	12,5
5	16	14	10

For h = 1-3 the three states share the seats; after this the seats are given away in this order:

B, C, B, A, C, B, C, A. C.

Now notice that before we can assign another seat, for house size h = 13, we need to add another row to this table. You can practice by computing a few more rows and assigning seats up to house size h = 20. Note that were we using D'Hondt's method, the order of assigning seats would B, C, A, and then continuing as in the string above. Note that while we might be interested only in the house size of 11, the table easily allows us to see what happens for h less than 11 and without adding any more rows, h up to 13. However, if we needed to assign 435 seats we might prefer to use the rounding rule divisor method associated with Jefferson's method.

You should complete the table below as far as possible using Adams' Method:

	B	C	A
Pop:	80	70	50
Row 1	Infinity	Infinity	Infinity
2	80	70	50
3			
4			
5			

Note we assign to the infinity in Row 1, arising because the population numbers are to be "divided by 0." We take this to mean that we assign each state one seat at the start when the house size is big enough, otherwise a tie-breaking system must be used. Note the curiously different way that seats are distributed as compared with Jefferson's method. We can see clearly that Jefferson's method tends to give seats earlier to large states while Adams's method tends to give seats earlier to small states. Of course, for many house sizes and population "vectors" we get the same apportionment using both methods.

Next, consider Webster's method. The completed table for carrying out Webster's method as a "table" method appears below. The rows are generated by dividing the populations of the states by 1, 3, 5, 7, etc. While these numbers don't arise directly from the "theory," they are used because of numerical convenience, and the mathematical observation that the theoretical values will not change in relative size when they are all multiplied by a constant.

	B	C	A
Pop:	80	70	50
Row 1	80	70	50
2	26.67	23.33	16.67
3	16	14	10
4	11.43	10	7.43
5	8.89	7.78	5.56

Note that in giving away the seats one at a time, when one reaches 10, there is a tie for who gets the next seat. We can think of this as having 2 house sizes where the seats are shared. Webster's method generally gives different apportionments from Huntington-Hill, which is the current method for apportioning seats to the House of Representatives. There is some controversy over whether or not Huntington-Hill is the best choice. Some people argue that within the divisor methods it is biased towards small states. Others argue that this bias is not really with Huntington-Hill but the requirement that small states get at least one seat as required by the constitution. What do you think?

I have not indicated the actual tables for this example using Dean or Huntington-Hill. You may want to practice these two methods also by computing the required tables and assigning seats. Be careful that there are enough rows in the table to carry out the assignment of seats for a house size of size h. Sometimes one has to go down another row of the table not yet computed before going across a row in the table to assign the next seat but forgets to add the extra row which would properly show how to assign the next seat.

On the web page of the US Census Bureau (Department of Commerce) you can find lots of interesting information and tables about the apportionment problem in the US and its history.

How does one choose among the 5 different "table" (divisor) methods? One can do so on the basis of either "absolute" differences or relative differences involving the number of representatives per person, the people per representative in comparisons or related measure between pairs of states . If one believes that relative differences are what really matter than one must choose the Huntington-Hill method. Dean's method is best for the absolute difference of people per representative while Webster's method is best for the absolute difference of representatives per person.

Reference:

M. Balinski and H. Young, Fair Representation, 2nd edition), Brookings Institution, Washington, 2001.

Note this book is a "classic" and is divided into two parts. The first is more or less expository writing about the apportionment question in the US and its history and the second half gives some of the details of a mathematical treatment.

Another important book, which emphasizes the point of view that arises from the European parliaments' version of the apportionment problem, where each political party need need be automatically assigned one seat in parliament is:

P. di Cortona, C. Manzi, A. Pennisi, F. Ricca, and B. Simeone, Evaluation and Optimization of Electoral Systems, SIAM, Philadelphia, 1999.

This book contains a discussion of the fact that D'Hondt tends to reward large parties at the expense of small parties. This can be viewed as a "positive" feature if it tends to bring about more stable coalition governments than might otherwise be the case. In the US, most people would argue that Jefferson's method is not appropriate because it rewards large states at the expense of smaller states and it will often violate the "quota" axiom with real data.

My two apportionment Feature Column articles for the American Mathematical Society are combined here:

http://rangevoting.org/MalkApport.html

This appears on the site that Warren Smith maintains to promote the Range Voting system of conducting elections. The articles can also be accessed directly from the AMS Feature Column site.

Also look at Paul Edelman's work arguing for a somewhat different approach to apportionment than "tradition" in using a divisor or Hamilton's method:

http://scholarship.law.berkeley.edu/cgi/viewcontent.cgi?article=1432&context=californialawreview

http://ecademy.agnesscott.edu/~jwiseman/old/mat325S08/stuff/edelmanApportionment.pdf

Here is an expository article about Edelman's work on what has come to be called the Minimal Total Deviation Method for apportionment:

http://law.vanderbilt.edu/alumni/lawyer-vol37num1/math.html

The intriguing issue here is how much work with basic ideas about relative versus absolute differences, fractions, percentages, inequalities and fairness come up from the extremely simple to state apportionment question.

When the Supreme Court was considering apportionment cases in the 1990's some of the government's arguments in favor of the Huntington-Hill method were prepared by Lawrence Ernst. Ernst has a background in statistics. This paper, which deals with what was behind the work in his briefs to the Court is fascinating.

http://www.bls.gov/ore/pdf/st940450.pdf