Mathematical Theory of Elections

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TABLE 1 shows the results of several elections in New York State. The data presented seem typical of American elections, perhaps unnoteworthy. In each of these elections, however, the candidate many people thought least likely to win was the winner! Why were the results of these elections so different from the results that many anticipated? In attempting to explain such situations, one might take a second look at the data, and study the political views of the candidates. One interesting common aspect apparent from the data in TABLE 1 is that unlike most American elections, there were three "major" candidates seeking office in these cases. Furthermore, it turns out that in each of these elections two of the candidates had similar political views, while the third candidate had views quite different from the other two. Perhaps in three-candidate elections, where the candidates have this pattern of political views, an unintuitive outcome is not really so surprising.

In order to investigate this question, let us briefly review the procedure used to elect candidates for nearly all major offices in America. Each voter casts a vote for his/her favorite candidate. The votes are counted, and the candidate with the largest number of votes is declared the winner. (Throughout our discussion here, we will assume that we are dealing with elections where the numbers of voters is so large that a tie vote among two or more candidates is so improbable that we can disregard this possibility.)

An election procedure can be separated into two components. The first component is the type of ballot used. Among the different type ballots available are those where the voter votes for only his/her favorite candidate; where the voter ranks the candidates in the order first choice, second choice, etc.; and where the voter not only ranks the candidates in order but also indicates, for example, how much more than his/her second choice he/she prefers his/her first choice. Another possibility allows the voter to cast votes for as many candidates as the voter "approves." In nearly all American elections, only favorite-choice ballots are used. The second component of the election system consists of choosing a vote-counting procedure based on the choice of ballot. The plurality system (i.e., the winner is the candidate with the largest number of votes) is almost exclusively employed.

Surely, you may ask, there is nothing wrong with the preceding procedure? Isn't this what democracy is about? Allowing each voter to select his or her favorite candidate and then choosing as winner the candidate with the largest number of votes. Doesn't this procedure select the choice of the people? Perhaps after taking a look at TABLE 2 you will not be so certain. TABLE 2 shows the possible outcomes in a three-candidate election. The data show that the winner received just more than one-third of the votes. Is it fair for A_3 to win this election? If the usual method of

TABLE 1.

1969 (Mayor, New York City)		
Lindsay	964,844	(Winner)
Prococino	813,316	("Expected" winner)
Marchi	538,404	
1970 (Senator, New York)		
Goodell	1,434,472	
Ottinger	2,171,232	("Expected" winner)
Buckley	2,288,190	(Winner)

conducting elections is not fair under all circumstances, then why is it used? As previously mentioned, most elections are contested between precisely two office seekers. In this case, one candidate will get a majority (not merely a plurality) of the votes cast. By majority candidate in an election, we mean a candidate who gets at least one more than one-half of the total votes cast. The distinction between obtaining a majority and a plurality is crucial in debates concerning the democratic nature of elections. If a candidate receives a majority of the votes cast, regardless of the number of candidates, that candidate's claim to be the victor in a democratic framework is secure. Unfortunately, if there are three or more candidates, there is no guarantee that one candidate will get a majority of the votes cast.

If winning by a majority is critical, why not proceed as follows in an election with more than two candidates? If no candidate gets a majority of the votes cast, eliminate all candidates except the top two vote-getters, and then hold a second election between these two candidates. One of these candidates must obtain a majority and can be declared the winner. This procedure is called the runoff method of election. To investigate this election method further, let us introduce some convenient ideas and notation. Let us denote the n candidates seeking office by $A_1, \ldots, A_n$. Each voter will be assumed able to *rank order* the candidates, that is, choose the candidate who is his/her favorite, next favorite, etc. A voter may be indifferent among two or more candidates. In FIGURE 1(a), we show the method a voter might use to indicate his/her preference among the candidates $A_1, \ldots, A_4$. The higher up on the arrow a candidate appears will mean that the voter prefers this candidate to all candidates further below on the arrow. When a voter is indifferent between two or more candidates, this is shown by placing the candidates at the same level on the arrow (FIG. 1(b)). Again, for simplicity, we will assume in what follows that no voter is ever indifferent between candidates. Diagrams such as those in Figure 1 are called *preference schedules.* An interesting problem in combinatorics is to determine the number of different preference schedules when there are n candidates in the case where no voter is indifferent, as well as in the case where indifference is allowed. The election method described previously can now be described and implemented for the election

TABLE 2.

Candidate A_1	Candidate A_2	Candidate A_3
x votes (x, a positive integer)	$x + 1$ votes	$x + 2$ votes

illustrated in FIGURE 2. Here, 55 voters cast their votes for 6 (of the 120) different preference schedules that can be constructed for 5 candidates when indifference doesn't occur. When plurality voting is applied to this election, A_1 wins, but does not get a majority; thus, we might choose to apply the runoff method. Note that this can be carried out without the voters physically going back to the polls, since all the preference information of the voters is contained in the schedules in FIGURE 2. (Of course, preferences of voters may change with time due to advertising, political blunders of the candidates, etc. Again, for simplicity we will discount this phenomenon). For the schedules in FIGURE 2, since in round 1 A_1 and A_2 are the largest first-place vote-getters, A_3, A_4, A_5 are eliminated, and in round 2 voters vote for whichever of A_1 and A_2 they prefer more. This method results in the election of A_2, a different winner than using the plurality method. Having thought of the runoff system, it may occur to you that eliminating all but the highest two vote-getters may eliminate popular "middle-level" candidates prematurely. To eliminate this objection, let us eliminate only the lowest vote-getter in each round. When this is done for the election in FIGURE 2, in round 1 A_5 is eliminated, in round 2 A_4 is eliminated, in

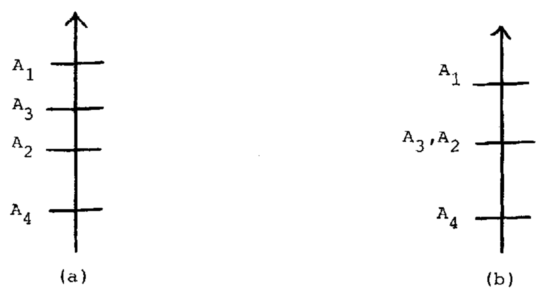
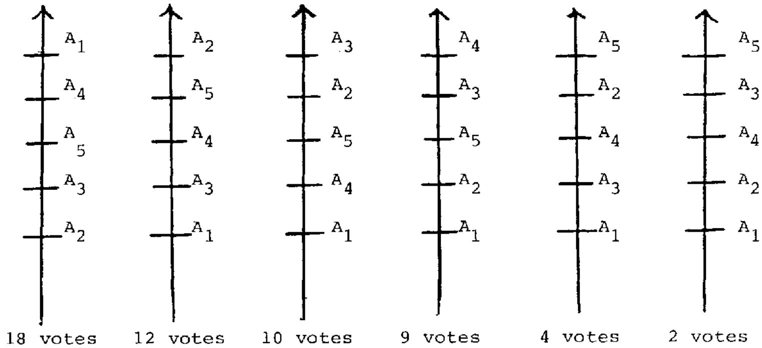


FIGURE 1.

round 3 A_2 is eliminated, resulting in the final round that A_3 wins, even though in the first election this candidate only got 10 (of 55) first-place votes. This method is known as sequential runoff. It is related to a method invented by British reformer Thomas Hare (1806–1891), known as the Hare method or single transferable vote. The Hare method is used in Australia and Ireland when a single seat is to be filled. A modification of this method is also used when more than one seat is to be filled in New York City school board elections. In this latter case, where many seats are to be filled it is designed to promote proportional representation. This refers to trying to have the proportion of candidates with certain characteristics reflect the portion of the electorate with these same characteristics.

Returning to our discussion of the election of FIGURE 2, since a candidate may stand relatively high, though not at the top of many preference schedules, one might try to design a system based on "points" that gives credit to a candidate for how high on a preference schedule a candidate appears. This method is generally credited to Jean-Charles de Borda (1733–1799). It works in one variant as follows: For each preference schedule, a candidate A_i will get as many points as the number of



(a)

Method 1 (Plurality voting)

$A_1 = 18$ $A_2 = 12$ $A_3 = 10$ $A_4 = 9$ $A_5 = 6$

Method 2 (Run off)

Round 1	$A_1 = 18$	$A_2 = 12$	$A_3 = 10$	$A_4 = 9$	$A_6 = 6$
Round 2	$A_2 = 18$	$A_2 = 37$			

Method 3 (Sequential Run off)

Round 1	$A_1 = 18$	$A_2 = 12$	$A_3 = 10$	$A_4 = 9$	$A_5 = 6$
Round 2	$A_1 = 18$	$A_2 = 16$	$A_3 = 12$	$A_4 = 9$	
Round 3	$A_1 = 18$	$A_2 = 16$	$A_3 = 21$		
Round 4	$A_1 = 18$	$A_3 = 37$			

Method 4 (Borda Count)

$A_1 = 4(18) + 0(12) + 0(10) + 0(9) + 0(3) + 0(2) = 72$

$A_2 = 0(18) + 4(12) + 3(10) + 1(9) + 3(4) + 1(2) = 101$

$A_3 = 1(18) + 1(12) + 4(10) + 3(9) + 1(4) + 3(2) = 107$

$A_4 = 3(18) + 2(12) + 1(10) + 4(9) + 2(4) + 2(2) = 136$

$A_5 = 2(18) + 3(12) + 2(10) + 2(9) + 4(4) + 4(2) = 134$

Method 5 (Concorcet)

A_1 versus A_5	$A_1 = 18$	$A_5 = 37$
A_2 versus A_5	$A_2 = 22$	$A_5 = 33$
A_3 versus A_5	$A_3 = 19$	$A_5 = 36$
A_4 versus A_5	$A_4 = 27$	$A_5 = 28$

(b)

FIGURE 2.

candidates below A_i on the preference schedule. For example, the leftmost preference schedule in FIGURE 2 (i.e., the one receiving 18 votes) results in 4 points for A_1, 3 points for A_4, 2 points for A_5, and 1 point for A_3. Of course, since 18 persons voted for this schedule, we weigh these points by 18. Thus, for the preference schedule in FIGURE 2 that got 12 votes, A_2 gets $4(12) = 48$ points, A_5 gets $3(12) = 36$ points, A_4 gets $2(12) = 24$ points, A_3 gets $1(12) = 12$ points, and A_1 gets $0(12) = 0$ points. Using the Borda count, A_1, A_2, A_3, A_4, and A_5 receive 72, 101, 107, 136, and 134 points, respectively—the winner being A_4. One clear problem with the Borda count is that it encourages voters not to be sincere when producing their preference schedules, since by changing the relative position of their on-first-place choice, they can increase the chance of their first-place choice winning (see [3], [5]). It is a nice exercise to show how examples of this phenomenon occur. More specifically, could the 18 voters who votes for A_1 as their first choice in FIGURE 2, alter the remainder of their preference schedule from the one shown, so that A_4 no longer is the Borda count winner, but A_1 wins instead?

Clearly, the different approaches discussed so far show the important role played by determining which of two candidates wins when placed in a two-way race against

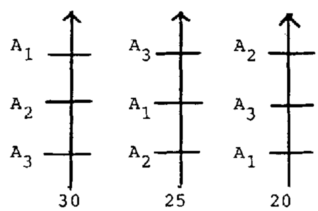


FIGURE 3.

each other. One can base an election system based on this idea. The system, first suggested by Marie-Jean-Antoine-Nicolas de Caritat, Marquis de Condorcet (1743–1794), is to declare as the winner of the election that candidate, if there is one, who can beat all the other candidates in a two-way race. For the election in FIGURE 2, this method would result in A_5 as the election winner.

FIGURE 2(b) summarizes the arithmetic calculations for these five election methods and shows the rather disturbing fact that the 5 methods yield 5 different winners! Which candidate should win? Which method should be used? Certainly in this example, A_5, the Condorcet winner, is an appealing choice. After all, what more could one require than that a candidate beat all others in two-way races? The "rub" with the Condorcet method, unlike the other four, is that it is not guaranteed to produce a winner. This fact was already known to Condorcet, and can be illustrated using the election in FIGURE 3, where A_1 beats A_2 55 to 20, A_2 beats A_3 50 to 25, and A_3 beats A_1 45 to 30! Thus, the relation R "beat in a two-way race," is not transitive, since aRb, bRc does not imply aRc. Note that this is despite the fact that we have assumed throughout that an individual can make transitive judgments among the alternative candidates. This is implicit in the preference schedule notation we have developed. For a discussion of the issue of transitivity, see [10]. Unhappily, the

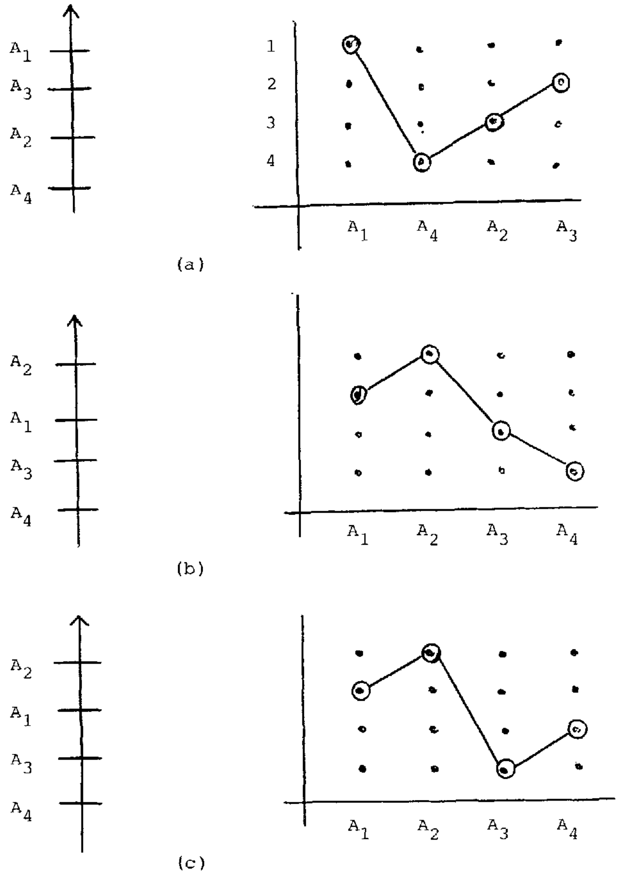


FIGURE 4.

Condorcet method, despite its overwhelming appeal when it is decisive, may not decide a winner. (See [2], [4], [6] for extensive ramifications.)

In order to study the Condorcet method of election more carefully, we will introduce a method due to Duncan Black [4] of "graphing" preference schedules. An $n \times n$ array of dots is constructed and the vertical axis labeled $1, 2, \ldots, n$ and the horizontal axis, with the candidates names, in any permutation. Two of the $n!$ possible ways to label the horizontal axis are illustrated in FIGURE 4. Given a preference schedule, we can draw a graph of it as follows: Locate the candidate name on the horizontal axis, and place a small circle in this column about the dot in the row that the voter uses on the preference schedule to rank this candidate. These dots are then joined by broken line segments. In FIGURE 4(b), the preference schedule has a "single-peaked" representation, while the preference schedules in FIGURE 4(a), (c) are not single-peaked. A *collection* of preference schedules is called single-peaked if for some listing of the candidates along the horizontal axis (recall there are $n!$ orders to try), all the preference schedules are simultaneously single-peaked. Intuitively, if a collection of schedules is single-peaked, one can think of the candidates as being in a left-to-right ("liberal-to-conservative") continuum along the horizontal axis. Duncan Black has shown that if there are an odd number of voters that produce a collection of single-peaked preference schedules, then the Condorcet method will produce a ranking of the candidates (i.e., intransitivity can't occur). This fact is not as attractive as it might seem at first glance, however, since for large numbers of candidates the total number of preference schedules is $n!$, while the size of a maximum set of a single-peaked schedule is 2^{n-1}. Furthermore, the conditions a set of schedules must obey to be part of a single-peaked set are very stringent. To see this, consider a maximum set of single-peaked preference schedules with n candidates. This is illustrated in FIGURE 5 for $n = 3$. Each such schedule can be extended uniquely to a single-peaked schedule for $(n + 1)$ candidates, as shown in FIGURE 6. What has been done to go from n to $n + 1$ is to place an additional row of dots under the existing rows and then to add a column of dots on the left. The preference schedules for n are extended to $n + 1$ by joining their left endpoints to X (shown dotted in FIG. 6).

By symmetry one can add the reflection of all these schedules in a vertical mirror shown in FIGURE 6. This gives 2^{n-1} additional schedules (not shown) through the bottom right dot or $2(2^{n-1}) = 2^n$ schedules for $(n + 1)$ candidates. Using a slight extension of this reasoning, one can show that the number of single-peaked preferences going through the points of each of the first two rows of diagrams like those in FIGURE 6 is given by the binomial coefficients! Thus, for $n = 4$, say, in a maximal single-peaked set of preference schedules at most 3 of the schedules can list a particular candidate in first place, and this can be true for at most two of the candidates, while in fact voters might produce 6 such schedules for each candidate. Thus, unless extraordinary conditions of homogeneity exist among a collection of voters, their chance of producing a single-peaked collection of preference schedules is unlikely. Results of this sort have been extended by Catherine Murphy [13]. Another approach to the Condorcet method is to compute the probability of the voting paradox (i.e., intransitive preference for society) occurring. For a large number of voters and 3 alternatives this probability is .080 (see [2] for details).

It is worth mentioning that all of the methods under discussion can be used not

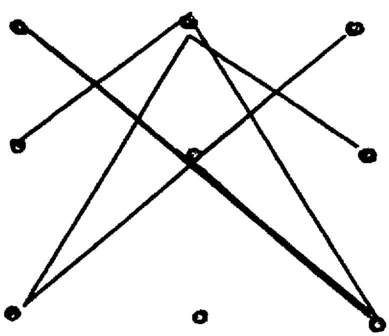


FIGURE 5.

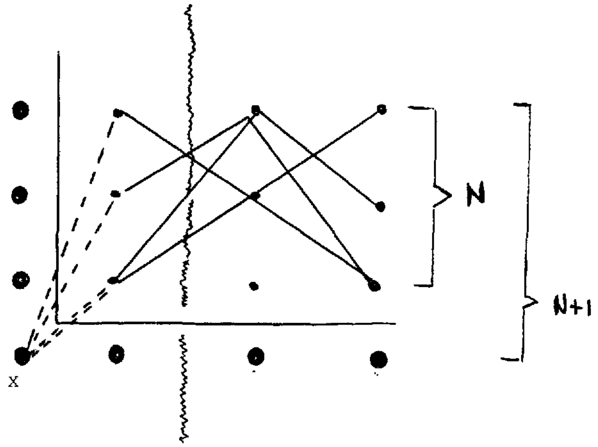


FIGURE 6.

only to produce a winner in the election but also to rank all the candidates. Thus, instead of ranking candidates, the same concepts can be used to obtain ranks for competing economic policies or options of any sort. It was in this framework that Kenneth Arrow did his pioneering work in about 1950. This work subsequently won him the Nobel Prize for economics, and gave impetus to the modern theory of social choice. Arrow reasoned that rather than looking for fair election (decision) methods, as had been done previously, one should list axiomatically those features of an election method that would make it fair. Hopefully, one could then determine election methods that met all the desired fairness criteria. Among the fairness criteria Arrow studied were the following.

(a) There should be no dictator (i.e., society chooses the ranking of a particular individual regardless of the input of all other voters).
(b) If every voter prefers option A_i to option A_j, then society should prefer A_i to A_j.
(c) The decision procedure used should not encourage voters to dissemble about their preferences.

In all, Arrow listed ([1]) five fairness criteria for any social decision method. He then proved the startling theorem that there was no election procedure that involved three or more candidates that met all five of his intuitively reasonable fairness conditions. Arrow's theorem had major ramifications within economics, political science, and mathematical political science. Much research has been done on extending the theorem and finding ways "around" it.

Recent work in social choice theory is particularly valuable for debunking the claim that mathematics is worthless as a tool for studying problems in the social sciences. In this instance, mathematics, political science, and economics have profited from these investigations. As a recent example of applied mathematics, Brams and Fishburn's work on approval voting [2] is excellent.

The mathematical theory of elections offers a dramatic example of how analytic reasoning can be exploited to obtain incisive insight into phenomena that affect everyday life. It is regretable that so small a part of the public has been made aware of mathematics' important role in understanding elections.

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