CONCEPTS

1 Limits and Continuity

1. **The Limit Laws**
   If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then
   - $\lim_{x\to c}[f(x) + g(x)] = L + M$
   - $\lim_{x\to c}[f(x) - g(x)] = L - M$
   - $\lim_{x\to c}[f(x) \cdot g(x)] = L \cdot M$
   - $\lim_{x\to c}[f(x)/g(x)] = L/M$, if defined
   - $\lim_{x\to c}[k \cdot f(x)] = k \cdot L$
   - $\lim_{x\to c}[f(x)]^{r/s} = L^{r/s}$, if defined

   Note: The Limit Laws apply to limits at infinity.
2. A limit fails to exist at a point if the function *jumps* around that point, *oscillates* around that point or approaches *infinity* around that point.
3. **The Sandwich Theorem**
   Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at c itself. Suppose also that $\lim_{x\to c} g(x) = L = \lim_{x\to c} h(x)$. Then $\lim_{x\to c} f(x) = L$.

   Note: The Sandwich Theorem applies to limits at infinity.
4. $\lim_{x\to 0} \frac{\sin x}{x} = 1$
5. A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$.
6. A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$.
7. A function f is **continuous** at an interior point c of its domain if $\lim_{x\to c} f(x) = f(c)$.
8. Types of discontinuities: *removable, jump, infinite, oscillating.*
9. **The Intermediate Value Theorem**
   A function $y = f(x)$ that is continuous on an interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. That is, if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(x_0)$ for some x_0 in $[a, b]$.

2 Derivatives

10. The **slope** of the curve $y = f(x)$ at the point $(x_0, f(x_0))$ is the number $m = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$. This number is also known as the **derivative** of f at x_0.

11. The **derivative** of f with respect to x is the function f' whose value at x is $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$, provided the limit exists.

12. A function fails to be differentiable at a point if there is *a corner, a cusp, a vertical tangent, or a discontinuity* at that point.

13. If a body's **position** at time t is $s(t)$, then

    - $v(t) = \frac{ds}{dt}$ is its **velocity**;
    - $a(t) = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$ is its **acceleration**.

    Also, **speed** is the absolute value of the velocity.

14. **Differentiation Rules**
    Let f and g be differentiable functions, and c a constant.

    - [power] $(x^n)' = nx^{n-1}$
    - [sum] $(f+g)'(x) = f'(x) + g'(x)$
    - [difference] $(f-g)'(x) = f'(x) - g'(x)$
    - [constant multiple] $(cf)'(x) = cf'(x)$
    - [product] $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$
    - [quotient] $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$
    - [chain] $(f \circ g)'(x) = f'(g(x))\, g'(x)$

15. **Derivatives of Trigonometric Functions**

    - $\frac{d}{dx} \sin x = \cos x$
    - $\frac{d}{dx} \tan x = \sec^2 x$
    - $\frac{d}{dx} \sec x = \sec x \tan x$
    - $\frac{d}{dx} \cos x = -\sin x$
    - $\frac{d}{dx} \cot x = -\csc^2 x$
    - $\frac{d}{dx} \csc x = -\csc x \cot x$

16. When solving a **related rates** problem, remember to

    (a) start by drawing a picture and name the variables and constants,
    (b) write down what you are given (the known quantities) and what you are asked to find (the unknown quantities),
    (c) write down an equation relating the known quantities and the unknown quantities,
    (d) differentiate the equation you obtained in part (c) with respect to time, and solve for what you were asked to find.

17. **The Extreme Value Theorem**
    If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value M and an absolute minimum value m in $[a, b]$.

18. An interior point c is a **critical point** of f if either $f'(c) = 0$ or $f'(c)$ is undefined.

19. The absolute extreme values of a continuous function defined on a closed interval are attained either at a critical point or at an endpoint.

20. **The Mean Value Theorem**
If f is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b), then there is at least one point c in (a, b) at which $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

21. Corollaries of the Mean Value Theorem

    (a) If $f'(x) = 0$ on an open interval (a, b), then $f(x) = C$ on (a, b), for a constant C.

    (b) If $f'(x) = g'(x)$ on an open interval (a, b), then $f(x) = g(x) + C$ on (a, b), for a constant C.

22. **First Derivative Test for Monotonic Functions** (*Increasing/Decreasing Test*)
Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

    - If $f'(x) > 0$ on $[a, b]$, then f is increasing on $[a, b]$.
    - If $f'(x) < 0$ on $[a, b]$, then f is decreasing on $[a, b]$.

23. **First Derivative Test for Local Extrema**
Suppose that c is a critical point of a continuous function f, differentiable near c. Moving across c from left to right,

    - If f' changes from negative to positive at c, then f has a local minimum at c.
    - If f' changes from positive to negative at c, then f has a local maximum at c.
    - If f' does not change sign at c, then f has no local extremum at c.

24. **Second Derivative Test for Concavity** (*Concavity Test*)
Suppose that f is twice differentiable on an interval I.

    - If $f''(x) > 0$ on I, then f is concave up over I.
    - If $f''(x) < 0$ on I, then f is concave down over I.

25. **Second Derivative Test for Local Extrema**
Suppose that f'' is continuous on an open interval that contains c.

    - If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
    - If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.
    - If $f'(c) = 0$ and $f''(c) = 0$, then the test is inconclusive.

26. **Strategy for solving optimization problems**

    (a) Draw a picture and label all variables and constants.

    (b) Write down every equation relating the known and the unknown quantities.

    (c) Express the unknown quantity as a *function* of the known ones. Find the *domain* of this function.

    (d) Find the absolute maximum or absolute minimum value of the function you obtained in Step (c). (Make sure you answer the question the problem asks!)

3 Integrals

27. A function $F(x)$ is an **antiderivative** of $f(x)$ if $F'(x) = f(x)$. The set of all antiderivatives of f is called the **indefinite integral** of f and is denoted by $\int f(x)\ dx$.

28. Let $y = f(x)$ be a nonnegative function on the interval $[a, b]$. Using n rectangles, the area A under $y = f(x)$ is approximated by $A \approx f(c_1)\Delta x + f(c_2)\Delta x + \ldots + f(c_n)\Delta x = \sum_{k=1}^{n} f(c_k)\Delta x$, where c_k is in the kth subinterval. Here, $\Delta x = \dfrac{b-a}{n}$. The exact value of the area is given by the **definite integral** $\int_a^b f(x)\ dx$.

29. **Integral of the Power Function** $\int x^n\ dx = \dfrac{x^{n+1}}{n+1} + C$

30. **Integrals of Trigonometric Functions**
   - $\int \cos x\ dx = \sin x + C$
   - $\int \sin x\ dx = -\cos x + C$
   - $\int \sec^2 x\ dx = \tan x + C$
   - $\int \csc^2 x\ dx = -\cot x + C$
   - $\int \sec x \tan x\ dx = \sec x + C$
   - $\int \csc x \cot x\ dx = -\csc x + C$

31. **Properties of the Definite Integral**
   Let f and g be integrable functions, and k a constant.
   - [order of integration] $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$
   - [zero width interval] $\int_a^a f(x)\ dx = 0$
   - [constant multiple] $\int_a^b kf(x)\ dx = k\int_a^b f(x)\ dx$
   - [sum and difference] $\int_a^b [f(x) \pm g(x)]\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$
   - [additivity] $\int_a^b f(x)\ dx + \int_b^c f(x)\ dx = \int_a^c f(x)\ dx$
   - [max-min inequality] If m is the minimum and M is the maximum of f on $[a, b]$, then
   $$m(b-a) \leq \int_a^b f(x)\ dx \leq M(b-a).$$
   - [domination]
     - If $f(x) \geq g(x)$ on $[a, b]$, then $\int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx$.
     - If $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\ dx \geq 0$.

32. **The Fundamental Theorem of Calculus**

Part 1 If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\ dt$ is continuous on $[a, b]$, differentiable on (a, b), and

$$F'(x) = \frac{d}{dx} \int_a^x f(t)\ dt = f(x).$$

Part 2 If f is continuous on $[a, b]$ and $F'(x) = f(x)$ on $[a, b]$ (that is, F is an antiderivative of f), then

$$\int_a^b f(x)\ dx = \Big[F(x)\Big]_a^b = F(b) - F(a).$$

33. **The Substitution Rule**
If $u = g(x)$ is differentiable and f is continuous on the range of g, then

$$\int f(\underbrace{g(x)}_{u})\ \underbrace{g'(x)\ dx}_{du} = \int f(u)\ du.$$

34. **The Substitution Rule for Definite Integrals**
If g' is continuous on $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(\underbrace{g(x)}_{u})\ \underbrace{g'(x)\ dx}_{du} = \int_{g(a)}^{g(b)} f(u)\ du.$$

35. **Area between curves**
If f and g are continuous and $f(x) \geq g(x)$ on $[a, b]$, then the area between $y = f(x)$ and $y = g(x)$ from a to b is

$$A = \int_a^b [f(x) - g(x)]\ dx.$$