The Time Constant of an RC Circuit

1 Objectives

1. To determine the time constant of an RC Circuit, and
2. To determine the capacitance of an unknown capacitor.

2 Introduction

What the heck is a capacitor? It's one of the three passive circuit elements: the resistor (which we've already met), the capacitor (which stores energy in electric fields), and the inductor (which stores energy in magnetic fields, and is the main subject a few weeks from now). Also known as condensers, capacitors store energy an electric field by separating positive and negative charges on opposing terminals known as "plates" - which may or may not actually be plate-like. Capacitors trace their lineage to the middle of the Eighteenth century and the invention of the Leyden jar. For forty years, the biggest names in natural philosophy spent time in the study of charge storage and improving capacitors, including Volta (for whom we name the volt) and Franklin (of American Founding Father fame).

In this lab, we will study the behavior of capacitors while we add or remove charge from the plates. Although we won't touch on it in this lab, RC circuits form the foundation of the modern electronics that underlie a whole host of technologies, including radio, computers, environmental and medical sensors, and - not least of all - the touch screens in cell phones and the touch pads of laptops.

3 Theory

We normally write Ohm's Law for the resistor as $V(t) = I(t)R$. But what is that current $I(t)$? In the circuits here (see Figure 1), it's just the rate at which charge passes through the resistor. Since all the charge that runs through the resistor ends up going from or to the capacitor, $q(t)$, it must also be the case that

$$I(t) = \pm \frac{\mathrm{d}q(t)}{\mathrm{d}t} \ ,$$

where $q(t)$ is the time dependent charge on the capacitor, with the sign depending on whether the charge on the capacitor is increasing or decreasing. For this circuit, we could instead

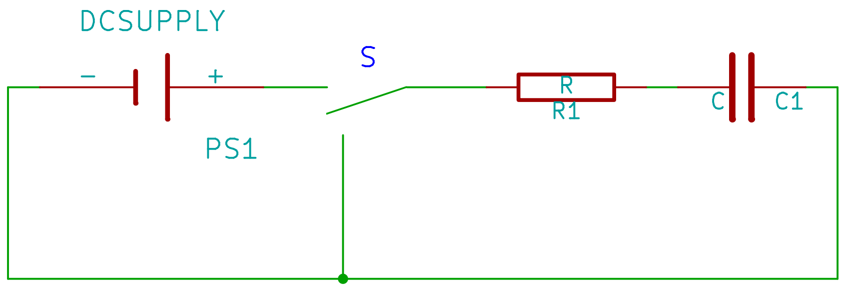


Figure 1: The switching circuit used to discuss charging and discharging a capacitor.

write Ohm's Law in the form

$$\frac{\mathrm{d}q(t)}{\mathrm{d}t} = \pm\frac{V(t)}{R} \ .$$

In words, a resistor is a passive device where the applied voltage causes charge to flow *through* the device, while a capacitor is a passive device where the applied voltage causes charge storage *in* the device; in equation form

$$q(t) = CV(t) \ .$$

The constant C is called the capacitance, and is measured in Farads (named for Michael Faraday, whose research output we will learn about later this semester) where $\mathrm{F} = {}^{\mathrm{C}}/_{\mathrm{V}}$.

Since capacitors are used to store charge, we must find a way to change the charge state on the capacitor. Let's discuss the circuit in Figure 1. This circuit has three states: 1. The steady state, where the switch is open, no current flows in the resistor, and the charge state on the capacitor is constant, 2. the "charging" state, where the battery or power supply is connected to the capacitor and adds charge to the capacitor, and 3. the "discharging" state, where the battery is disconnected, the two plates of the capacitor are connected to each other through the resistor, which removes charge from the capacitor.

We can analyze the dynamic states of the circuit using Kirchoff's Rules[1]:

1. The Loop rule (energy conservation) requires the sum of all voltages drops around a closed loop to vanish, and

2. The Junction rule (charge conservation) requires the sum of all currents into a junction to vanish.

Let's apply these rules, first to the discharging and then to the charging state:

1. Discharging: In the discharging case, the current will flow off the capacitor in a counter clockwise direction (why?). When we close that switch, Kirchoff's rules become

$$\begin{aligned} V_c - V_R &= 0 \\ -\frac{\mathrm{d}q(t)}{\mathrm{d}t} - \frac{q(t)}{RC} &= 0 \ . \end{aligned}$$

[1]As long as the frequencies aren't so high that we can't use the *lumped element* approximation; talk to your instructor if you want to know more.

Again, the rate is negative, because the capacitor is discharging, not charging. The method of solution for this equation is given in Appendix A:

$$V_c(t) = V_R(t) = V_s e^{-t/RC} \ ,$$

assuming that the initial voltage across the capacitor is V_s. This “discharge curve” is plotted in Figure 2.

2. Charging: In the charging case, the current flows clockwise from the battery (at voltage V_s), through the resistor (at V_R), and across the capacitor V_c; we want to solve for V_c as a function of time. Kirchoff’s Rules tell us

$$\begin{aligned} V_s - V_R - V_c &= 0 \\ V_s - IR - \frac{q}{C} \end{aligned}$$

But here, the current through the resistor depends on the rate at which charge is changing on the capacitor; they have the same sign here since the charge on the capacitor is increasing

$$\begin{aligned} V_s - \frac{\mathrm{d}q(t)}{\mathrm{d}t} R - \frac{q(t)}{C} &= 0 \\ \frac{\mathrm{d}q(t)}{\mathrm{d}t} + \frac{q(t)}{RC} &= \frac{V_s}{R} \ . \end{aligned}$$

We can solve this differential equation too for the time dependent voltage profile across the capacitor; see Appendix A. If we start with a completely discharged capacitor, the voltages across the resistors and capacitors vary as

$$\begin{aligned} V_c(t) &= V_s \left(1 - e^{-t/RC}\right) \\ V_R(t) &= V_s e^{-t/RC} \ . \end{aligned}$$

As the capacitor charges, the voltage across - and hence the charge *on* - the capacitor rises exponentially, while the voltage across - and hence the current *through* - the resistor will fall exponentially; see Figure 2.

The quantity RC - which appears in the argument of the exponential - is known as the *time constant* of the system; it has units of time (hence the name), and determines the time interval over which voltages, charges, and currents change in the circuit. The time constant can be tuned by modifying either R or C. In practice, more resistor than capacitor values are commercially available, so it’s usually easier to tune the resistor values.

Given a known resistance we can measure the time constant of the RC circuit, and algebraically determine the capacitance C. When discharging the capacitor,

$$V_c(t) = V_s e^{-t/RC} \ .$$

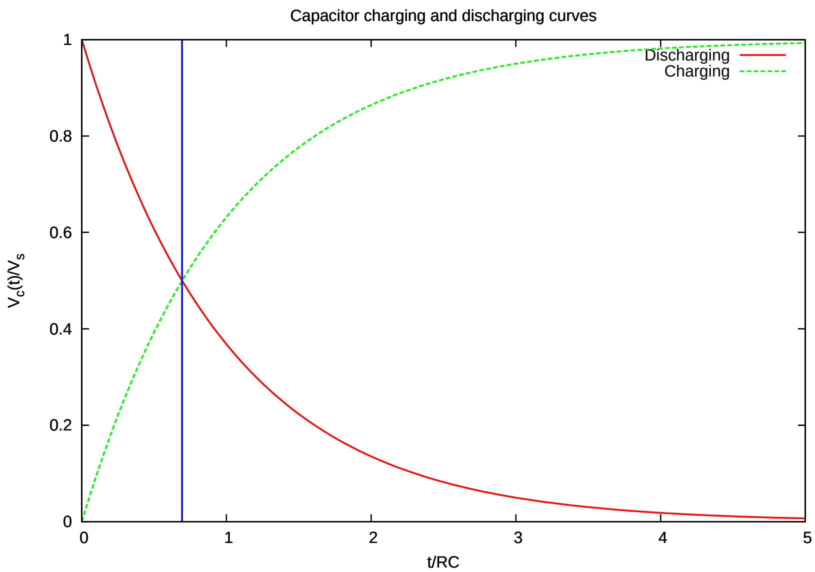


Figure 2: The capacitor charging and discharging curves. The vertical blue line is the "half life" point of the charging and discharging timeline.

We can easily measure and use the half-life $T_{1/2}$ of the discharge: $T_{1/2}$ is the time it takes for the voltage to fall by half. Substituting into the above equations and solving for $T_{1/2}$:

$$\begin{aligned}\frac{V_s}{2} &= V_s e^{-T_{1/2}/RC} \\ \ln\frac{1}{2} &= -\frac{T_{1/2}}{RC} \\ T_{1/2} &= (C\ln 2)\,R\ .\end{aligned}$$

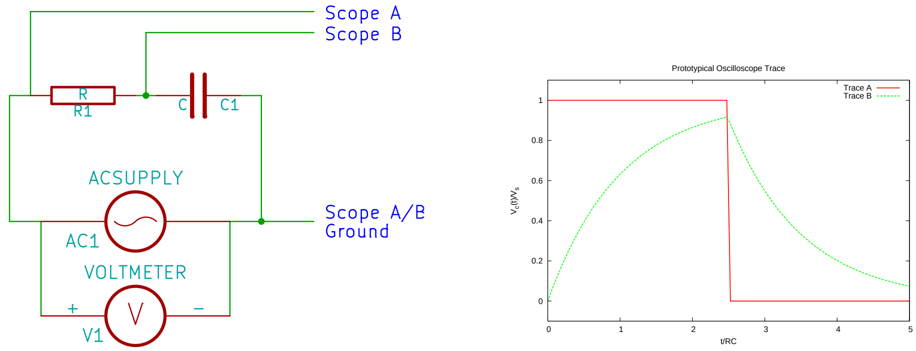
If we allow R to vary, then this equation defines a line with abscissa R, ordinate $T_{1/2}$, and slope $C\ln 2$. If we measure multiple $(R, T_{1/2})$ pairs, we can plot those data points, fit them to a line, and extract the slope - and hence the capacitance.

4 Procedures

You should receive an oscilloscope, a function generator, a multimeter, one unknown capacitor, a decade capacitance box, and a decade resistance box.

4.1 Time Constant of an RC Circuit

In this part of the experiment, instead of a DC voltage and a mechanical switch, we apply a *square wave signal* to the capacitor as shown in Figure 3. An ideal square wave has two values: high and low (here V_s and 0), and it switches between them instantaneously. The capacitor will charge when the voltage of the square wave is V_s; the capacitor will discharge when the voltage of the square wave is zero. The oscilloscope traces of the charging and discharging of the capacitor are also shown in Figure 3.



(a) Measurement Circuit (b) Oscilloscope Display

Figure 3: The left-hand figure is the circuit used to measure the time constant of an RC circuit, while the right-hand figure shows the Oscilloscope traces.

If the period of the square wave T_s is much less than the time constant $\tau = RC$ ($T_s \ll \tau$), then the capacitor will start discharging before it has sufficient time to acquire the maximum charge q_0, making measurement of the time constant difficult. What will this look like on the oscilloscope? If T_s is much more than τ ($T_s \gg \tau$), you will not be able to fit both the charging and discharging portions of the capacitor voltage on the oscilloscope trace at the same time. Why?

1. Construct the circuit shown in Figure 3a.
2. Perform the initial turn-on procedure for the oscilloscope (from the Oscilloscope Lab). To start, set both channels to 0.2 V/div. Make sure that the ground of both traces is at the same position on the screen.
3. Start by setting the frequency of the square wave to about 100 Hz and the output voltage to 0.5 V.
4. Now we have to set R and C. Adjust R to about 8 kΩ, and C to about 0.1 μF. What is the time constant of this combination?
5. Adjust the frequency of the square wave, and the time and voltage settings of the oscilloscope until you obtain traces similar to Figure 3b. Measure and record R, C, f, V, $T_{1/2}$, and the oscilloscope settings.
6. Measure the voltage of the decay curve above ground at a number of points using the oscilloscope cursors; similarly, measure the height of the charging curve at a number of points. You will use these to map out the charging/discharging curves, and fit for the time constant.
7. Repeat for a second set of R and C.

4.2 Determine the Capacitance

Here, we'll use the same techniques we just did to determine the value of an unknown capacitor.

1. Replace the known capacitor with the unknown capacitor in the circuit.
2. Set the resistance to about $4\,\mathrm{k}\Omega$, and make the necessary adjustments to the oscilloscope settings to again obtain the appropriate display on the oscilloscope.
3. Measure and record R, f, V, the oscilloscope settings, and $T_{1/2}$.
4. Repeat for four or five values of R; you need enough points to fit a line through your $(R, T_{1/2})$ pairs.
5. Use your multimeter to directly measure the capacitance, for comparison to your extracted value.[2]

A Derivation of Solutions

The differential equations in Section 3 are the simplest of all differential equations to solve: first order, linear equations with constant coefficients. Here, we'll solve a slightly more complicated equation (a first order linear equation with non-constant coefficients) in complete generality, and use this solution to find the solutions we're interested in. Consider the following differential equation:

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x} + P(x)y(x) = Q(x)\ .$$

This equation looks very much like the equations we found in Section 3 using Kirchoff's Rules. We can solve this in the following way. First, we'll multiply both sides of this equation by a new function, $M(x)$, and we'll first solve for $M(x)$.

$$M(x)\frac{\mathrm{d}y(x)}{\mathrm{d}x} + M(x)P(x)y(x) = M(x)Q(x)\ .$$

Let's now assume that the left hand side is the expansion of the product rule of $M(x)y(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(M(x)y(x)\right) = M(x)Q(x)\ .$$

What must be true for this to hold? Well,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(M(x)y(x)\right) = M(x)\frac{\mathrm{d}y(x)}{\mathrm{d}x} + \frac{\mathrm{d}M(x)}{\mathrm{d}x}y(x) = M(x)\frac{\mathrm{d}y(x)}{\mathrm{d}x} + M(x)P(x)y(x)\ .$$

This is true if and only if

$$\frac{\mathrm{d}M(x)}{\mathrm{d}x} = M(x)P(x)\ .$$

[2] The multimeter uses an automated version of exactly this procedure to calculate C, based on a time constant measurement across a precision internal resistor.

This is a "trivial" problem to solve: just separate and integrate:

$$M(x) = e^{\int P(x)\mathrm{d}x + c} = e^c e^{\int P(x)\mathrm{d}x} = c' e^{\int P(x)\mathrm{d}x} \ ,$$

where c is the constant of integration for this indefinite integral. Now, go a few lines back up the page:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(M(x)y(x)\right) = M(x)Q(x) \ .$$

We can solve this one trivially, too, just by integrating; let's make this a definite integral between x_0 and x. We get the left hand side from the Fundamental Theorem of Calculus

$$M(x)y(x) - M(x_0)y(x_0) = \int_{x_0}^{x} M(x)Q(x)\mathrm{d}x \ ,$$

where, since this is an indefinite integral, requires an integration constant c. Finally, then, we can solve the general problem:

$$y(x) = M(x)^{-1}\left(M(x_0)y(x_0) + \int_{x_0}^{x} M(x)Q(x)\mathrm{d}x\right) \ .$$

Note that the integration constant in the definition of $M(x)$ cancels here, so it doesn't really matter what it was; let's just choose $c' = 1$.

In the case of the charging RC circuit with zero initial charge, $x = t$, $y(x) = q(t)$, $x_0 = 0$, $y(x_0) = 0$, $P(t) = 1/RC$, and $Q(t) = V_s/R$. Integrating gives us $M(t) = e^{t/RC}$, giving us the solution:

$$\begin{aligned} q(t) &= e^{-t/RC}\int_0^t e^{t/RC}\frac{V_s}{R}\mathrm{d}t \\ &= \frac{V_s}{R}RCe^{-t/RC}\left(e^{t/RC} - 1\right) \\ &= V_s C\left(1 - e^{-t/RC}\right) \ , \end{aligned}$$

where we've used the integration constant to get the $t = 0$ value right (here, no charge on the capacitor). Dividing both sides by C gives us the voltage across the capacitor at time t

$$V_c(t) = V_s\left(1 - e^{-t/RC}\right) \ .$$

In the case of the discharging circuit, we have

$$q(t) = V_s C e^{-t/RC} \ ,$$

where we assume the voltage across the capacitor at $t = 0$ is V_s. Again, divide by C to get the voltage profile

$$V_c(t) = V_s e^{-t/RC} \ .$$

Pre-Lab Exercises

Answer these questions as instructed on Blackboard; make sure to submit them before your lab session!

1. What is the time constant for an RC circuit with $R = 45\,\mathrm{k\Omega}$ and $C = 1.2\,\mu\mathrm{F}$?

2. If it takes $5\,\mu\mathrm{s}$ for a capacitor to charge to half the battery voltage, through a $10\,\mathrm{k\Omega}$ resistor, what is the capacitance C?

3. A given RC circuit charges through a particular resistor R, and capacitor, C. If I double the capacitance, what happens to the time constant? What if I double the resistance?

Post-Lab Exercises

1. Determine the time constant of the RC circuit with known capacitance. Do this by plotting your data of time versus the amplitude of the signal. You should fit this data to an exponential, and extract τ. There are a number of ways to do this: (a) use a mathematics package capable of sophisticated statistical analysis, such as R, Octave, Matlab, Mathematica, Root, etc., or (b) using a spreadsheet, fit the data using an exponential trendline analysis. Alternatively, you can transform your spreadsheet data into a linear data set, and do a linear trendline analysis. Estimate the uncertainty of this measured time constant. You should also estimate the time constant from the measured $T_{1/2}$. Do these multiple experimental extractions agree within your estimated uncertainty? Do they agree with the theoretical expectation given your measured R and C?

2. Determine the unknown capacitance. Plot your $(R, T_{1/2})$ data pairs, and fit the data to a line. Extract the capacitance from the fitted line parameters. Estimate the uncertainty of this extracted capacitance. You can also estimate the capacitance from the individual $(R, T_{1/2})$ pairs; estimate your uncertainties. Do these estimates agree with the line fit estimate?

3. Discuss briefly whether you have met the objectives of the lab exercises.