Alternating Current RC Circuits

1 Objectives

1. To understand the voltage/current phase behavior of RC circuits under applied alternating current voltages, and

2. To understand the current amplitude behavior of RC circuits under applied alternating current voltages.

2 Introduction

While you have studied the behavior of RC circuits under direct current conditions, very few interesting circuits have purely direct currents and constant applied voltages. All productive or interesting circuits operate under alternating current conditions - think computers, radios (including cell phones), etc.

In a previous lab[1] you studied the behavior of the RC circuit under constant applied (or DC) voltages. Here, you will study the behavior of the same circuit under sinusoidally alternating applied (or AC) voltages (see Figure 1).
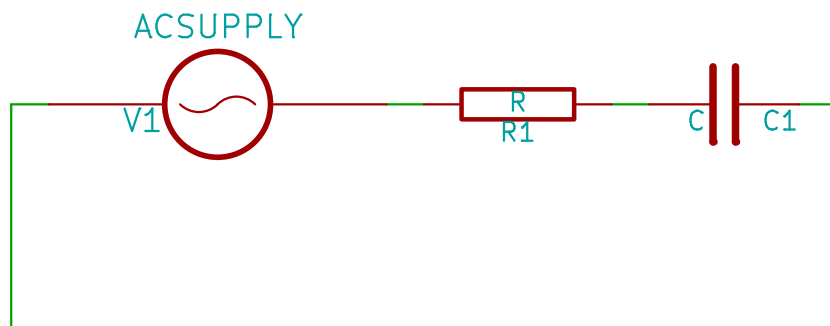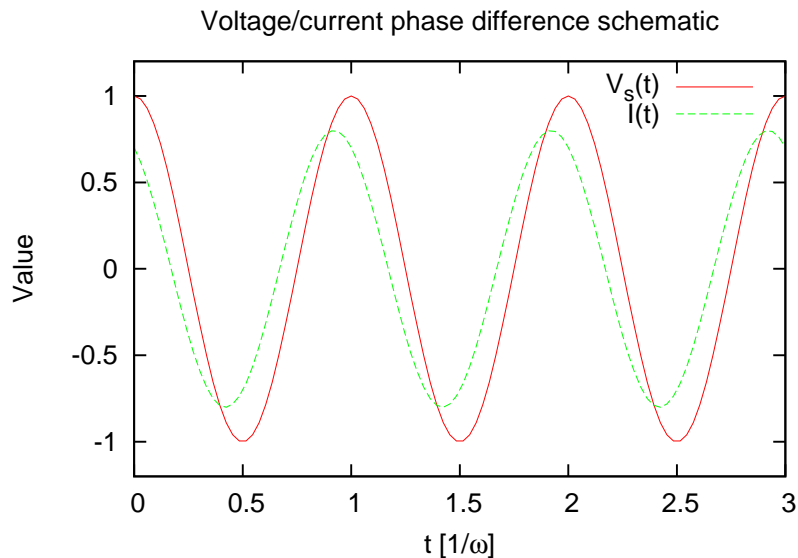

Figure 1: The RC circuit.

Figure 2: A schematic of the phase difference between the applied voltage $V(t)$ and the derived current $I(t)$.

3 Theory

Let's begin analyzing this circuit the same way you analyzed the DC RC circuit, via Kirchoff's Rules. As before, you'll find

$$V_s(t) - V_R(t) - V_C(t) = 0 \ .$$

Again, just as in the DC case,

$$V_C(t) = \frac{q(t)}{C} \qquad\qquad V_R(t) = I(t)R \qquad\qquad I(t) = \frac{\mathrm{d}q(t)}{\mathrm{d}t} \ ,$$

leading to the differential equation

$$\frac{\mathrm{d}q(t)}{\mathrm{d}t} + \frac{q(t)}{RC} = \frac{V_s(t)}{R} \ ,$$

which has the general solution

$$q(t) = e^{-t/RC} \left[q(t_0) e^{t_0/RC} + \int_{t_0}^{t} e^{t/RC} \frac{V_s(t)}{R} \mathrm{d}t \right] \ .$$

If $V_s(t)$ is allowed to be any old arbitrary function, you're stuck. But getting stuck takes all the fun out of your work, so of course, you can't allow it to be an arbitrary function: let's focus on sinusoids. This is a class of broadly useful functions - they're what come out of the wall, for instance. But most importantly, they are highly amenable to mathematical manipulation and analysis, due to Fourier's theorem: any well behaved function can be

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[1] *The Time Constant of an RC Circuit*

decomposed into a (potentially infinite) sum of sinusoids of all possible frequencies. Let's take

$$V_s(t) = V_s \cos \omega t .$$

In Appendix A we'll derive the full solution; here, you will require only the *steady state response*, which gives the time dependent charge as

$$q(t) = V_s C \frac{1}{\sqrt{1 + (RC\omega)^2}} \sin(\omega t + \phi) ,$$

where $\tan \phi = 1/RC\omega$. You know ϕ as a *phase constant*. When you calculate the current flow, $I(t)$, in this circuit, you will find a remarkable thing:

$$I(t) = \frac{V_s}{R} \frac{RC\omega}{\sqrt{1 + (RC\omega)^2}} \cos(\omega t + \phi) .$$

Because of the presence of the capcitor, the applied voltage $V_s(t)$ gives rise to a current which not only has a frequency dependent amplitude, but more importantly has a different phase than the source voltage; see Figure 2. Because *phi* is positive, the current rises slightly before the voltage: we say the current *leads* the voltage, or that the voltage *follows* the current.

In our teaching labs, we don't have the tools to measure the current profile and compare it directly to the applied voltage - remember, we only have ammeters, voltmeters, and oscilloscopes (which behave for most purposes like voltmeters). To use the oscilloscope to measure this phase difference, you must find a voltage that follows exactly in phase with the current ... and you have one of those: the voltage across the resistor. Measuring $V_R(t)$ and comparing with $V_s(t)$ allows us to measure ϕ.

$$V_s(t) = V_s \cos \omega t$$
$$V_R(t) = V_s \frac{RC\omega}{\sqrt{1 + (RC\omega)^2}} \cos(\omega t + \phi) = I(t)R$$
$$V_C(t) = V_s \frac{1}{\sqrt{1 + (RC\omega)^2}} \sin(\omega t + \phi) ,$$

There is another point of interest: the behavior of the system as a function of frequency. In the limit that the frequency goes to zero (that is a DC voltage), the steady state behavior of this system should look just like the DC system you studied previously: $V_R(t)$ should go to zero, $V_s(t)$ should equal the applied voltage. You should check these assertions. In the other extreme, where the frequency gets large, you probably have no *a priori* expectations. Plotting the behavior as a function of frequency (see Figure 3), you will find that the amplitude of $V_C(t)$ vanishes, while the amplitude of $V_R(t)$ goes to V_s, while the phase difference between the applied voltage and resulting current also vanishes. You can prove this by taking the limits of the voltage expressions when $\omega \to \infty$. In other words, the circuit acts like the capacitor isn't even there! Capacitors become transparent to currents at high frequency, and opaque to currents at very low frequencies. This is known as *filtering* behavior, and is the basis of most of the interesting behaviors in the analog electronics we all use everyday.
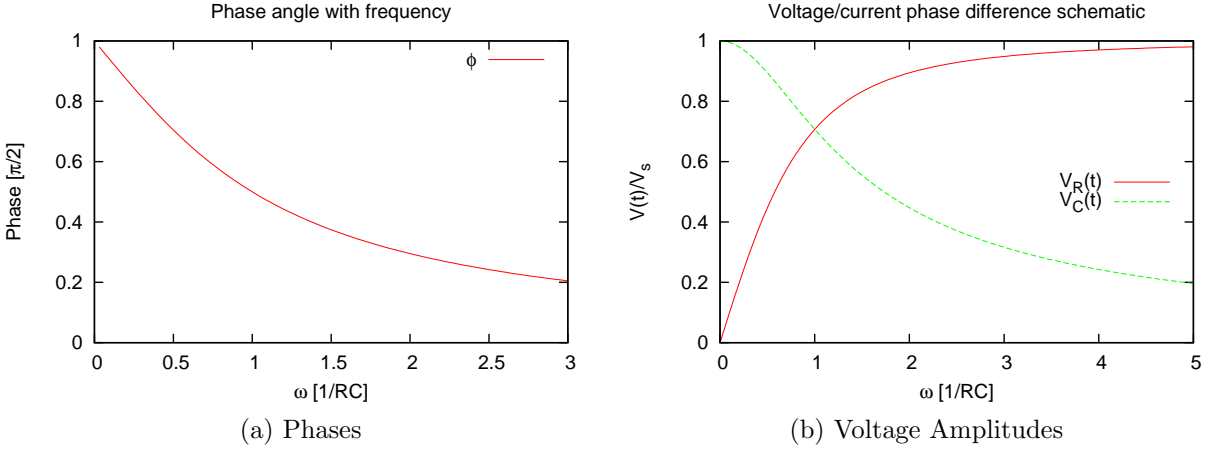
(a) Phases

(b) Voltage Amplitudes

Figure 3: The phase angle as a function of angular frequency, while the function amplitudes are displayed on the right. In both cases, the frequency is normalized in units of $1/RC$. The phase is normalized to $\pi/2$, while the amplitudes are normalized to V_s. Notice the limiting behavior when $\omega = 1/RC$.

There is another way to view the complexities of voltages and currents in AC RC circuits. Notice that the quantity $RC\omega$ is dimensionless; in other words, $1/C\omega$ has the units of resistance. It *isn't* a resistance (it's not a constant, for starters), but it has the same units, and some of the same properties. Let's define the quantity

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C} \; ,$$

which is called the *capacitive reactance* of the circuit. Next, define the *impedance Z*

$$Z^2 = R^2 + X_C^2 \; .$$

As it turns out, if you look only at the amplitudes of the current and applied voltages, they are related by

$$V_s = ZI \; ,$$

a sort of generalized version of Ohm's Law. Note further that

$$\tan\phi = \frac{1}{RC\omega} = \frac{X_C}{R} \; .$$

With these definitions, you can rewrite the voltage equations for the resistor and capacitor as

$$V_R(t) = V_s \frac{R}{Z} \cos(\omega t + \phi) = I(t)R$$

$$V_C(t) = V_s \frac{X_C}{Z} \sin(\omega t + \phi) \; .$$

By combining these with the definition of impedance above, you also find
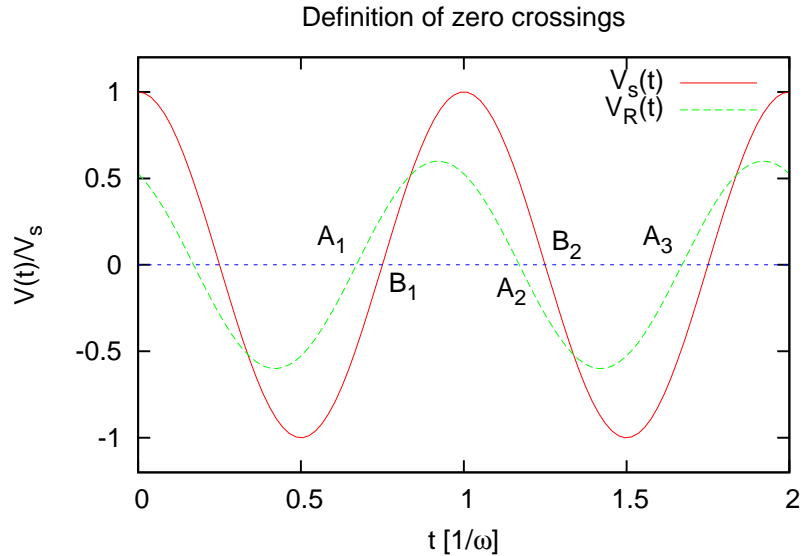
$$V_s^2 = V_R^2 + V_C^2 \; .$$

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Figure 4: Definition of the points on the oscilloscope curve called out in Step 4.

4 Procedures

You should receive two multimeters, an oscilloscope, a function generator, a decade resistance box, and a decade capacitor box.

1. First, let's select component values for testing. Choose a value for the capacitance between $0.06\,\mu\text{F}$ and $0.1\,\mu\text{F}$. Select a frequency between $300\,\text{Hz}$ and $600\,\text{Hz}$. Calculate X_C and choose a value for $R \approx 1.2 X_C$. Measure and record the values of R and C

2. Configure the circuit for testing shown in Figure 1. Insert one multimeter to record the AC current; except in the last two steps of the procedure, make sure the current remains constant throughout the experiment.

3. Using the other meter, record the frequency f, and the RMS AC voltages across the signal generator V_s, the resistor V_R, and the capacitor V_C.

4. Let's measure the phase shift between the current and applied voltage. Connect the oscilloscope so as to measure the voltage across the resistor and signal generator; make sure the negtive inputs share a common reference point. Make sure the two signal base-lines are centered with respect to the horizontal and vertical axes of the oscilloscope, and adjust the voltage and time scales so that slightly more than one cycle of both waveforms is visible. You should have a display that looks roughly like Figure 4. We're going to record the differences between the zero crossings, and calculate the phase from these differences. Record (at least!) $A_1 A_3$, $A_1 B_1$, and $B_1 A_2$ using the cursors.[2] Increase the frequency by 50%, and determine the phase shift again. Double the initial frequency, and repeat.

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[2] $A_2 B_2$ and $B_2 A_3$ are redundant with $A_1 B_1$, and $B_1 A_2$, and $A_1 A_2$ should equal $A_2 A_3$ if you have properly centered the sinusoid vertically.

5. Next, map out the amplitude of the current response. Without changing R and C, vary the frequency over, say, ten points, and record the frequency, RMS voltage V_s, and RMS current I at those points. Measure and record your observations of the amplitudes of V_s and V_R on the oscilloscope.

A   Derivation of Solutions

The differential equation for the AC RC circuit is given in Section 3. It has the general solution

$$q(t) = e^{-t/RC} \left[q(t_0) e^{t_0/RC} + \int_{t_0}^{t} e^{t/RC} \frac{V_s(t)}{R} \mathrm{d}t \right] .$$

If $V_s(t)$ is allowed to be any old arbitrary function, we're stuck. But getting stuck takes all the fun out of your work, so of course, we won't allow it to be an arbitrary function: we'll focus on sinusoids. This is a class of broadly useful functions - they're what come out of the wall, they're present in electromagnetic radiation, and they are highly amenable to mathematical manipulation and analysis. Let's take

$$V_s(t) = V_s \cos \omega t .$$

From your Physics I course, you should remember that V_s is the *amplitude* of the oscillation, while $\omega = 2\pi f$ is the *angular frequency*. With this applied voltage, we *can* perform the integration, which I leave as an exercise to the reader.[3]

Upon integrating and collecting terms, you will find a very complicated looking expression:

$$q(t) = q(t_0) e^{-(t-t_0)/RC} - e^{-(t-t_0)/RC} \frac{V_s}{R} \frac{1}{(1/RC)^2 + \omega^2} \left(\frac{1}{RC} \cos \omega t_0 + \omega \sin \omega t_0 \right)$$
$$+ \frac{V_s}{R} \frac{1}{(1/RC)^2 + \omega^2} \left(\frac{1}{RC} \cos \omega t + \omega \sin \omega t \right) .$$

Notice that the first line contains a decaying exponential dependence on time; wait long enough, and those terms all die off. This is called the *transient response* of the circuit, which comes from the initial charge on the capacitor and the initial action of turning on the function generator at time t_0. The second line has no exponential dependence, and is called the *steady state* response of the circuit. That's the part we are really interested in, and it looks fairly awful in this form. Let's clean it up some.

Consider the following expression

$$\alpha \cos \omega t + \omega \sin \omega t .$$

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[3]Hint:

$$\cos \omega t = \frac{e^{i\omega t} + e^{-i\omega t}}{2} .$$

Both terms have the same frequency, so we should be able to rewrite this as a single sinusoid, with a phase shift

$$A\sin(\omega t + \phi) \, .$$

Applying the trigonometric angle addition identities, we find

$$A\sin(\omega t + \phi) = A\sin\phi\cos\omega t + A\cos\phi\sin\omega t \, .$$

Equating terms in this expression with the first expression in the paragraph, you find

$$\alpha = A\sin\phi \qquad\qquad \omega = A\cos\phi \, .$$

Solving for A and ϕ, you should obtain

$$A = \sqrt{\alpha^2 + \omega^2} \qquad\qquad \tan\phi = \frac{\alpha}{\omega} \, .$$

Substituting back into the steady state response in the previous paragraph, where $\alpha = 1/RC$, we find

$$q(t)_{\text{ss}} = V_s C \frac{1}{\sqrt{1 + (RC\omega)^2}} \sin(\omega t + \phi) \, .$$

Pre-Lab Exercises

Answer these questions as instructed on Blackboard; make sure to submit them before your lab session!

1. Calculate the reactance of a $0.01\,\mu$F capacitor at a frequency of $250\,$Hz.

2. If an RC circuit has a $50\,\Omega$ resistor in series with a $1\,\mu$F capacitor, what will its impedance be at $500\,$Hz?

3. An RC circuit has a $5\,$kΩ resistor and a $1\,\mu$F capacitor. At what frequency will the current lead the voltage by $\pi/4$?

4. An RC circuit has a $5\,$kΩ resistor and a $1\,\mu$F capacitor. This circuit is driven by a $100\,$Hz sine wave with $1\,$V amplitude. What is the amplitude of the current in the circuit?

Post-Lab Exercises

1. From your measured resistance, capacitance, and frequency, determine the reactance and impedance of your circuit. Make sure to estimate your uncertainties. Determine the impedance experimentally via another method, taking care of the uncertaintites.Do you get the same results?

2. Estimate the uncertainties on the measured values of V_s, V_C, and V_R. Are the three values consistent with each other? Explain what you mean by "consistent".

3. Describe qualitatively what happens to your signals when you vary the frequency.

4. From your measurements in Step 4 of the procedure, determine the phase shift at each of the three measured frequencies, including an estimate of the uncertainty. How do these compare to the theoretical predictions?

5. Is your data from Step 5 consistent with the predictions of theory? Specifically, do the voltage and current amplitudes measured by oscilloscope and by multimeter match, within uncertainties, and do they comport with theoretical expectations?

6. Discuss briefly whether you have met the objectives of the lab exercises.