Gifts from Euler's Polyhedral Formula

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Introduction

In 1750, in a letter to Christian Goldbach, Euler wrote his friend that he had discovered a formula that related the number of vertices, faces, and edges of a polyhedron. As was characteristic of the period he did not give a formal discussion of what polyhedra his formula applied to, and he probably was thinking only of what today we would call convex polyhedra. Two years later he published two papers in which he discussed his polyhedral formula. However, the articles do not contain a rigorous proof.

With regard to the early history of the polyhedral formula it is sometimes claimed that the result was known to Descartes. However, this does not appear to be true. Rather, Descartes had proved a result from which Euler's result can be deduced but did not make that deduction. (Descartes' Theorem involves the sum of the "defects" at the vertices of a convex polyhedron adding to 720 degrees.) In light of all the work on geometry that was done in ancient Greece it seems surprising that the result was unknown in the ancient world, yet that is what appears to be the case. Many inquire in light of the fact that Euler invented graph theory in 1735 with his work on what is today called Euler circuits, why he did not use graph theory ideas or methods to try to investigate his newly discovered polyhedra formula. The reason appears to be that Euler did not think of polyhedra as graphs any more than Descartes did. When he looked at a "corner" of a polyhedron he did not see a vertex of a graph, he saw a "solid angle." In the notation that he uses to express his result, he refers to counting "solid angles."

The first rigorous proof of Euler's formula appears to have been given by Legendre. Today, when we think of Euler's formula as theorem in combinatorial geometry, Legendre's metrically based proof seems strange.

However, it was a natural approach for the time since Euler's result was but one beginning of a combinatorial approach to geometry. The extension of Euler's formula to plane connected graphs was done by Cauchy. Cauchy gives a very appealing argument for proving his extension, but by modern standards the "proof" is not rigorous. (His idea is to take the plane graph and triangulate all but the infinite face, while verifying that the quantity $C = V + F - E$ remains unchanged during the verification. He then shows that C remains unaltered as one performs a series of reductions which reduces the connected plane graph to a triangle (strictly speaking he did not consider the full collection of plane connected graphs, only those that arose from polyhedra). One can make Cauchy's approach rigorous using the concept of a "shelling" due to Peter Mani.

The history of what happened after Euler's announcement of the formula is rich indeed. It involves the study of the evolution of the concept of polyhedra to objects with holes and an attempt to be precise about the concept of what the face of a polyhedron is allowed to be. The ideas here not only lead to investigations of polyhedra by Cauchy and others, but to the material (via what today we would call the Euler characteristic) which lead to the birth of algebraic and point set topology.

The richness of the polyhedral formula is hinted at in the numerous proofs that can be given of this result, many of which have been collected by David Eppstein on his web site.

I now turn to the "gifts" from Euler's formula. Using Euler's formula what additional mathematics can be accomplished? The answer very much depends on how broad a net one casts. Here I will restrict my attention to some of the uses that Euler's formula has found in the areas of graph theory and geometry but I want to emphasize that this is the tip of the iceberg, and I will not even try to be complete in surveying its uses in graph theory and geometry.

Steinitz's Theorem

Along with Euler's theorem itself, no doubt the most important result concerning 3-dimensional convex polyhedra is the result generally referred to as Steinitz's Theorem. The hard part of the theorem concerns determining what condition(s) on a planar graph will guarantee that the graph can arise as the vertex-edge graph of some 3-dimensional bounded convex polyhedron. It will be convenient to use Branko Grünbaum's term d-polytope to mean the set which results from taking the convex hull of a finite set of points which yields a d-dimensional object (i.e. does not lie in any lower dimensional subspace of d-space.) Grünbaum is also responsible for translating the work of Steinitz into modern terminology, thereby making it accessible to other geometers. We will say a graph is d-polytopal if it is isomorphic to the vertex-edge graph of some d-polytope.

Theorem (Steinitz, as reformulated by Grünbaum and Motzkin)

A graph G is 3-polytopal if and only if it is planar and 3-connected.

Here 3-connected means that given any two vertices v and w of the graph there are at least 3 (simple) paths from v to w whose only vertices (or edges) in common are v and w.

One significance of Steinitz's Theorem is that when one studies the combinatorial properties of 3-polytopes one does not have to have the skill of being able to visualize them in three-dimensional space. Instead, one can work with a special class of planar objects.

Steinitz gave several proofs of his theorem and the circle of ideas involved in his proofs are explored in the proofs that Grünbaum gives in his seminal book Convex Polytopes. Subsequently, other routes to Steinitz's Theorem have been explored and these results are given in Ziegler [ ]. For our purposes we note that Euler's theorem is used in all of these approaches. Steinitz's Theorem has lead to an explosion of insights into such phenomena as Hamiltonian circuits on 3-polytopes and matchings of 3-polytopes. These results are indirectly a gift from Euler's formula.

Existence Theorems for Polyhedra

From the earliest mathematical writings up to the present there have been systematic attempts to find conditions on polyhedra which lead to a finite number of examples, or a simple classification theorem.

In Euclid's Elements there is a "proof" that there are 5 regular solids: solids whose faces are (convex) regular polygons and whose vertices are all alike.

However, using Euler's formula one can prove a bit more than this. Define a combinatorially regular solid to be one whose faces are all p-gons, q at a vertex. Since each edge of a 3-polytope has exactly two endpoints and lies on exactly two faces we can write: $pF = 2E$ and $qV = 2E$. Substitution in Euler's formula yields the diophantine equation $1/p + 1/q = 1/2 + 1/E$., where p and q each have to be at least 3. This equation has exactly 5 solutions, which correspond to the five regular solids known to the Greeks.

Over the centuries other classes of polyhedra to be looked at have been:

1. Regular polyhedra which allow non-convex regular faces.

2. Archimedean polyhedra which allow the faces to be regular polygons (perhaps of several different numbers of sides) but where the pattern of faces around each vertex is the same.

It turns out that there are 14 such 3-polytopes (in addition to the two infinite families, the prisms and anti-prisms), but to preserve the tradition of the 13 examples found by Archimedes (as described in the writings of Pappus) and Kepler, we use a different definition of Archimedean today. 3-polytopes with regular polygons as faces and whose symmetry group acts transitively on the vertices are known as Archimedean. Pappus makes no explicit mention of the prisms and anti-prisms but these families are explicitly mentioned in the work of Kepler.

3. The convex deltahedra, 3-polytopes with all faces equilateral triangles.

It turns out there are 8 of these.

4. Regular faced solids.

Norman Johnson raised the question of exactly which 3-polytopes have regular faces. He made a conjecture that there were 92 such solids (other than the prisms and anti-prisms). Eventually Zallgaller verified this result. These polytopes are now known as the Johnson solids.

5. Constant face vector polyhedra.

Consider k-valent 3-polytopes (k = 3, 4, or 5) for which the number of k-gons, p_k is equal to m, for each k that p_k is not zero. There are only a finite number of such polyhedra.

5. Regular polyhedra.

H. M. S. Coxeter and B. Grünbaum found expanded classes of regular polyhedra by relaxing the definition of what is consider to be a regular polygon. Coxeter for example allows "skew regular polygons" and Grünbaum allows even more general polygons. Eventually, Andreas Dress showed that with small corrections the list Grünbaum gave was complete.

6. Isosceles triangle 3-polytopes.

A collection of 3-polytopes that deserves more attention are those which have isosceles triangles for faces. An interesting special class are those whose faces are all congruent isosceles triangles.

In all of the situations above, typically one can use arguments based on Euler's formula to assist in getting insight into what is going on.

Next, consider k-valent 3-polytopes. We can obtain for a fixed k, a diophantine equation that such a polytope would have to obey. Here are the details in the case for k = 4. Since the polytope is 4-valent we have that 4V = 2E. We also have two equations that involve the faces of the polytope: first, the number of faces F is given by : $F = \Sigma p_k$ and second we have $\Sigma kp_k = 2E$. Substituting into Euler's formula we get the "Euler relation:"

$$p_3 = 8 + \Sigma(k-4)p_k \quad (*)$$

There are similar equations for 3-valent and 5-valent polyhedra but the thing noteworthy about the 3-valent and 4-valent cases is that there is a value of k for which the coefficient of p_k vanishes.

This situation has led to an area that has come to be known as "Eberhard Theorems." This is a collection of theorems that uses a supply of the faces that are "unrestricted" due to the zero coefficient to construct polyhedra with various nice properties.

An especially nice recent result in this area is the theorem of Dalyoung Jeong which guarantees the realization by a 4-valent 3-polytope having a "cut-through" Eulerian circuit of any face vector satisfying (*) using some choice for the unrestricted number of 4-gons! (Note that this theorem can be interpreted as saying something about the face vectors which can be achieved by the projections of knots in the plane.)

An important open question is to understand the situation regarding the existence of 5-valent 3-polytopes, since for these there can be no direct analog of Eberhard's Theorem. There is some work on this in a paper of J.C. Fisher.

There are also Eberhard type theorems that one can study concerning trees.

This has been the basis for some work on spanning trees in graphs which must have or can be guaranteed to lack spanning trees with 2-valent vertices.

Coloring problems

Heawood's well known theorem that planar graphs are 5-colorable has usually been proved using Euler's polyhedral formula, though there is a spectacular new proof of a generalization of Heawood's theorem by Carsten Thomassen. Remarkably Thomassen gives an inductive proof, where the induction hypothesis is cleverly chosen so that only the Jordan Curve Theorem and not Euler's formula is used. Euler's formula is also an important tool in both of the proofs of the 4-color theorem.

Computational geometry

Many results in the emerging field of computational geometry rely on Euler's formula.

Graph drawing and thickness

Graph drawing is the emerging part of graph theory concerned with "optimal" ways of drawing a graph in the plane. One topic of interest in graph drawing is the crossing number. The crossing number of a graph is the minimum number of edge crossings, (edges that meet at a vertex are not considered to cross) in any drawing of the graph in the plane. Many extensions of the crossing number concept have been defined and are being actively explored.

With regard to crossing numbers an essential tool for the study of planar graphs is the fact that if a graph has 3 or more vertices then it must satisfy:

$$E \leq 3V - 6.$$

Using this result a variety of planarity and crossing number results follow. In particular, one can use Euler's formula to show that K_5 and $K_{3,3}$ can not be planar. The thickness of a graph G is the minimum number of planar graphs on the same number of vertices as G whose union of edges form the edges of G. Not surprisingly, Euler's polyhedral formula is a crucial tool in studies of graph drawing problems and questions involving the thickness of graph.

Fullerenes

Euler's formula has been a valuable tool in examining the properties and existence of 3-valent 3-polytopal graphs with 12 pentagons and hexagons which have come to be known as the fullerenes. This collection of solids, of great importance to physicists and chemists, raises many questions of mathematical interest.

Further gifts

The emerging field of computational geometry, the area of mathematics and computer science concerned with the construction of geometric algorithms, makes extensive use of Euler's formula.

Euler's formula has had a rich history and offered a rich legacy of ideas that inspired growth in many parts of geometry and topology. There is little doubt that it will continue to offer up gifts in the future.

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