A Polygon Which Folds to a Convex and Non-Convex Deltahedron

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Recently, the appealing topic of folding polygons to obtain 3-dimensional polyhedra has gotten more attention in geometry classrooms. Many teachers now show students that there are 11 polygons which will fold to become a cube, each of the 11 being the union of edge-to-edge squares (polyominoes). Figure 1 shows the edges of a cube that must be cut to unfold the cube to a flat polygon. The dark edges shown on the cube include all of the vertices and do not contain a circuit, and, thus, constitute a spanning tree for the cube.

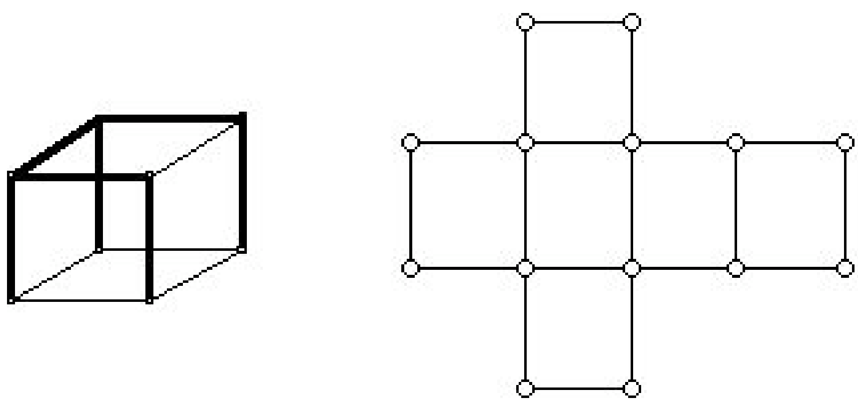

Figure 1

This note is an attempt to call attention to some of the subtle issues involved in folding polygons to polyhedra and unfolding polyhedra to get polygons. Deltahedra are 3-dimensional polyhedra all of whose faces are congruent equilateral triangles.

Shown in Figure 2 is a plane polygon which has been decomposed into equilateral triangles with internal diagonals. The polygon has 10 distinguished vertices, all of which lie on the boundary of the polygon.

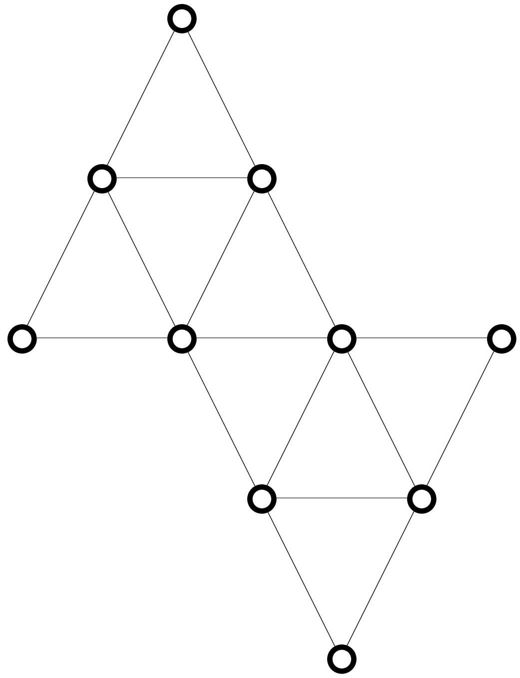

Figure 2

Figure 3 shows the same polygon with the vertices labeled with the first 10 letters of the alphabet - a to j. Note that the polygon has 7 internal edges, and can be thought of as a triangulated plane polygon.

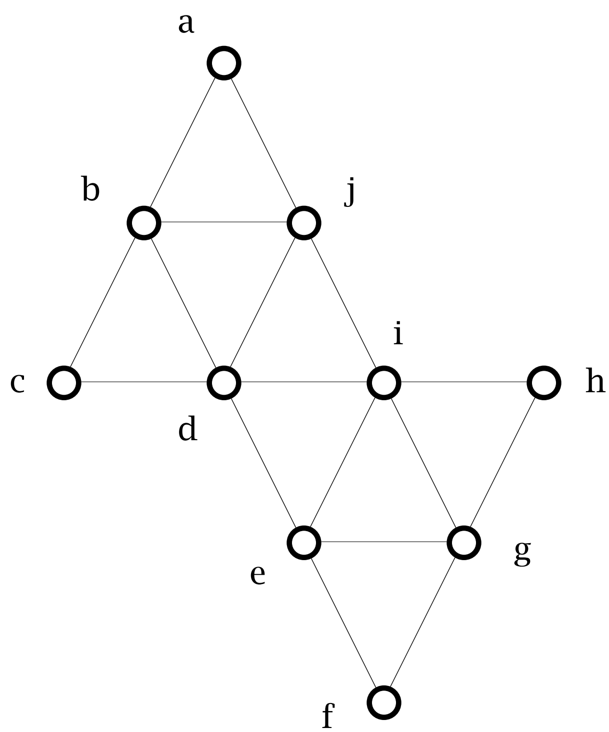

Figure 3

The angles at these vertices are:

Vertex	Angle
a	60
b	180
c	60
d	240
e	180
f	60
g	180
h	60
i	240
j	180

The sum of these angles is 1400 degrees, which is in accord with the fact that the sum of the interior angles of a 10-gon is (n-2)(180) for an n-gon. (To prove this, note that every simple n-sided polygon, convex or non-convex, can be decomposed into triangles using the existing vertices of the polygon.)

This polygon will fold into two polyhedra, one convex and the other non-convex! One of these two polyhedra is a regular octahedron R which has 6 vertices, 8 faces, and 12 edges. All six of the vertices of this polyhedron are 4-valent. (You can verify that Euler's polyhedral formula V + F - E = 2 holds.)

For the octahedron R we have: the following identifications (gluings) for the edges:

cd to de

cb to ef

ab to fg

ij to ih

aj to gh

This means that some pairs of vertices become identified:

a with g

b with f

c with e

h with j

The vertices d and i that occur at reflex angles are not paired with another vertex. This accounts for the final 6 vertices of the octahedron which is created. The 5 edges that result from identifying the 10 edges of the polygon form a spanning tree of this polyhedron which has 6 vertices. These 5 edges form a path between the two vertices d and i of the polyhedron.

At the vertices of R the angles from the original polygon obey:

ag: 60 + 180 = 240

bf: 180 + 60 = 240

ce: 60 + 180 = 240

hj: 60 + 180 = 240

d: 240

i: 240

This means that the defect (360-(sum of the face angles)) at each of the 6 vertices is 120. Since 6(120) is 720, Descartes' Theorem holds as it must for a convex polyhedron. Descartes' Theorem states that for any convex polyhedron the sum of the defects is 720 degrees. In fact, it holds more generally for any polyhedron which is topologically equivalent (homeomorphic) to a sphere.

The other polyhedron into which this polygon will fold is a non-convex octahedron R*. This octahedron has 6 vertices, 8 faces, and 12 edges. R* has two vertices which are 3-valent, two vertices which are 4-valent and two vertices which are 5-valent.

For the octahedron R* we have: the following identifications (gluings) for the edges:

ab and ih

aj and ij

bc and gh

cd and fg

de and ef

This means that some pairs of vertices become identified:

a and i

b and h

c and g

d and f

The two vertices e and j which occur at straight angles are not identified. This accounts for the final 6 vertices of the octahedron which is created. The 5 edges that result from identifying the 10 edges of the polygon form a spanning tree of this polyhedron which has 6 vertices. These 5 edges form a path between the two vertices e and j of the polyhedron.

At the vertices of R* the angles from the original polygon obey:

ai: 60 + 240 = 300

bh: 180 + 60 = 240

cg: 60 + 180 = 240

df: 240 + 60 = 300

e: 180

j: 180

This means that the defect (360-(sum of the face angles)) at two vertices is 60, at another two vertices is 120, and at two vertices is 180. This gives 120 + 240 + 360 which is 720 as a total defect.

The line segment from e to j is not an edge of R* and lies outside of R* which is non-convex. Figure 4 will give you some idea of what R* looks like.

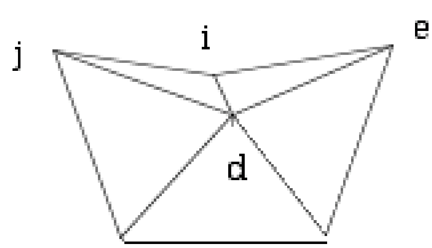

Figure 4

At first glance it may appear that the folding which gives rise to R* is a violation of Alexandrov's Theorem (see Appendix), because the gluing obeys the conditions of Alexandrov's Theorem. We have glued all of the boundary, we have not identified vertices where the angle sum is more than 360, and we have a result which is topologically equivalent (homeomorphic) to a sphere.

Question: Why does the existence of R* not violate Alexandrov's Theorem?

In order to see why it helps to look at Figure 5. Note that the polygon has two edges which lie along a common straight line.

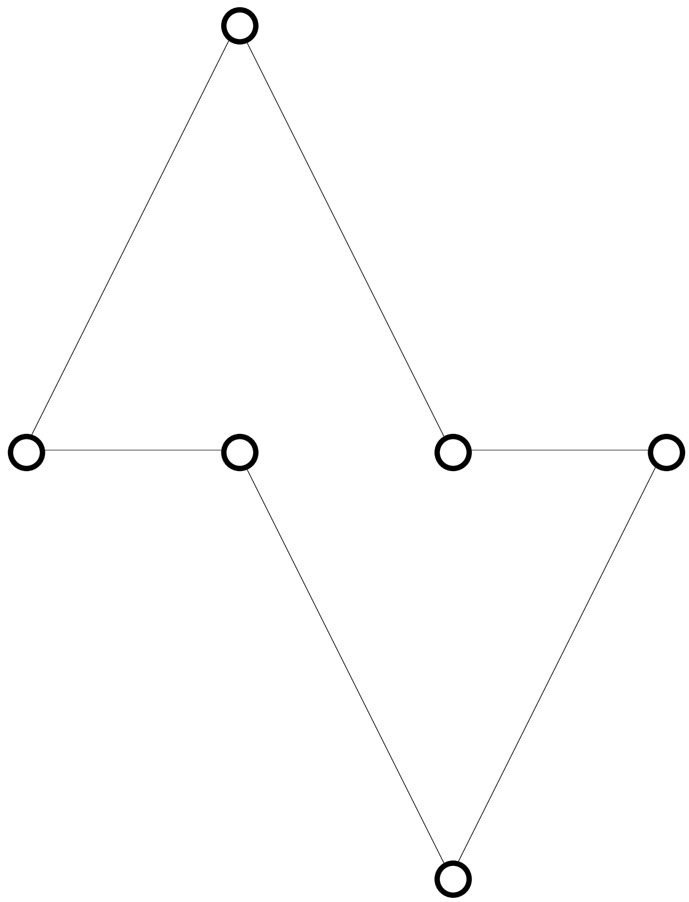

Figure 5

or its labeled version, Figure 6, which includes some additional vertices. As a polygon it has some vertices with straight angles at these vertices.

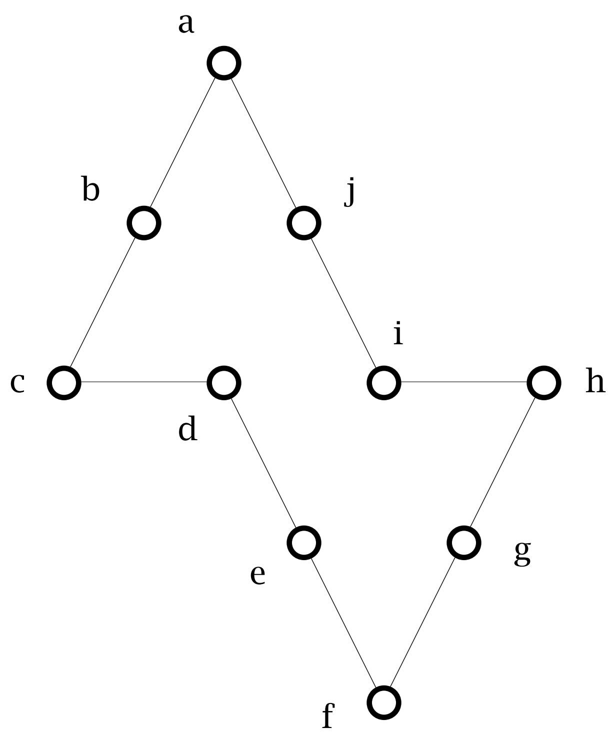

Figure 6

When we start with a polyhedron which is convex or non-convex and for which cutting along a spanning tree of edges will open up the polyhedron into a simple polygon, we can add to that polygon the edges of the polyhedron that we started with which were not cut. These edges appear as diagonals in the polygon we open up. For R and R* the polygon is that of Figure 1 (Figure 2 with labels). However, Alexandrov's Theorem does not say anything (explicitly) about the diagonals of the polygon we are attempting to fold up to form a polyhedron. It merely says that if we create a gluing obeying the rules, the result is a convex polyhedron or a double covered convex polygon.

The gluing described above gives rise to a convex polyhedron. However, it is not the polyhedron R*, which is not convex! By looking at Figure 4 can you visualize the convex polyhedron that it will fold to using the gluing in Figure 5? I was unable to but when I used a piece of paper in the shape of Figure 4, I found some clues. The resulting convex polyhedron "appeared" to be a triangular prism. The triangular prisms we commonly see have equilateral triangles for the two triangular faces and square or sometimes rectangles for the 4-gonal faces. However, Figure 4 with the gluing indicated in Figure 6 appeared to be a triangular prism where the two triangles are equilateral triangles and the three 4-gonal faces whose angles are 60 and 120 degrees. Thus, the 4-gons appeared to be the result of pasting together two equilateral triangles along a common edge. Whoops! It turns out that one can't get a triangular prism with three rhombuses (which are made of two equilateral triangles) and two equilateral triangles. There exists a triangular prism with two equilateral triangles in parallel planes and then two rhombuses which are unions of equilateral triangles; the fourth 4-gon is a square!

Thus, Alexandrov's Theorem shows that there is a convex polyhedron with the gluing used for R* but the exact combinatorial type and metrical properties of this convex polyhedron I have been unable to determine.

Appendix

Alexandrov's Theorem is named for Alexander Danilovich Alexandrov (1912-1999). There are a variety of spellings for all three parts of his name since it is being written using English language characters. Alexandrov was Russian and the important theorem he proved, often referred to as Alexandrov's Theorem, was originally published in Russian and only became know outside of Russia many years after its discovered. Alexandrov had already started publishing work about convex polyhedra as early as 1933. However, the result that concerns us, about folding a plane polygon, appears to have been done in the 1940's. The theorem can be stated in relatively elementary terms (which capture its intuitive essence) but the proof is very complex and belongs to the domain of differential geometry. Differential geometry, loosely speaking, is the branch of geometry which involves issues of "curvature."

Alexandrov's Theorem (Intuitive statement)

If a simple plane polygon is folded up (glued, zipped up) so that all of its boundary is "used up" the result is a convex polyhedron or a double covered convex polygon.

What precisely does folded up, glued, zipped up mean?

Without going into very technical detail, here are the ideas.

1. If the given simple plane polygon is P we want all of the points of the boundary to be matched with other parts of the boundary in such a way that all of the points are used up, though we allow individual points to be matched with themselves.

2. If many points of the boundary of P are matched up together at a point W the total angle at the points that are glued at W must be at most 360 degrees.

Comment: If a point Q lies in the interior of an edge of P then the angle at Q is 180 degrees. If we match Q and Q' which are interior points of different different edges of P together this allowed because the sum of the angles at the point where they are matched will be 360 degrees.

3. If gluing the boundary does not result in a double covered polygon then the result should be a "surface" which is topologically like a sphere (homeomorphic to a sphere).

Comment: If the gluing is not a double covered polygon the surface we get, which is "like that of a sphere" is in fact a convex polyhedron.

Comment: An especially interesting type of gluing is "perimeter halving." For any point T on the boundary of P there is a unique other point T* on the boundary of P which makes the lengths of the two paths from T to T8 along the boundary of P exactly equal in length. One way to glue P's boundary is to imagine there are zippers from T to T along these two paths, and to zip up the polygon using these zippers! The convex polyhedra that one can get from a polygon using perimeter halving gluing represent only a portion of those that can be obtained from more general gluings

Comment: When one cuts along the edges of a convex polyhedron and gets a "net" polygon from the polyhedron, Alexandrov's Theorem can be applied to this net as we have seen and there are many more convex polyhedra that can be gotten from this net other than the original one. We have also seen that even with edge to edge foldings we can get a polyhedron different from the one we started with.

References

E. Demaine and J. O'Rourke, Geometric Folding Algorithms, Cambridge U. Press, New York, 2007.

O'Rourke, J., Folding polygons to convex polyhedra, in Understanding Geometry for a Changing World, 71st Year Book, T. Craine and R. Rubenstein (eds), NCTM, Reston, 2009.

O'Rourke, J., How To Fold It, Cambridge U. Press, New York, 2011.

Shephard, G., Convex polytopes with convex nets, Mathematical Proceedings of the Cambridge Philosophical Society, 78 (1975) 389–403.