

Alternating Current RLC Circuits

1 Objectives

1. To understand the voltage/current phase behavior of RLC circuits under applied alternating current voltages,
2. To understand the current amplitude behavior of RLC circuits under applied alternating current voltages, and
3. To understand the phenomenon of resonance in RLC circuits.

2 Introduction

In previous labs¹ you studied the behavior of the RC and RL circuits under alternating applied (or AC) voltages. Here, you will study the behavior of a similar circuit containing series connected capacitor, inductor, and resistor. This is, quite reasonably, called an *RLC Circuit*; see Figure 1.

3 Theory

Once again, let's analyze this circuit using Kirchoff's Rules. As always, you find

$$V_s(t) - V_R(t) - V_L(t) - V_C(t) = 0 ,$$

¹Alternating Current RC Circuits and Alternating Current RL Circuits

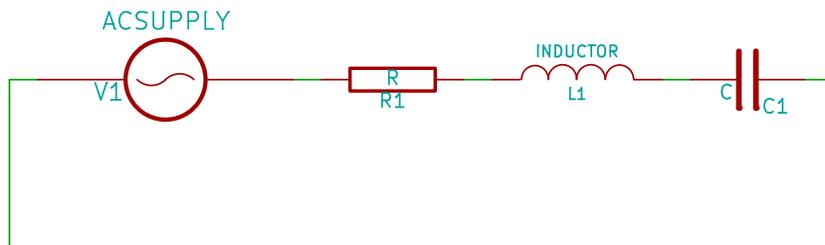


Figure 1: The RLC circuit.

leading to a differential equation we have not encountered in these labs before

$$\frac{d^2q(t)}{dt^2} + \frac{R}{L} \frac{dq(t)}{dt} + \frac{1}{LC}q(t) = V_s(t) ,$$

where $q(t)$ is the charge on the capacitor. We will not actually solve this equation, as the derivation is beyond the mathematics level of this course; however, in Appendix A we quote some important results. For a sinusoidally varying source voltage

$$V_s(t) = V_s \cos \omega t ,$$

we find the current is again out of phase, but this time, whether the current lags or leads the applied voltage depends on whether the inductive or capacitive reactances (both defined as before) dominate the behavior of the circuit at the driving voltage. Comparing the solutions in the Appendix with our differential equation here, matching coefficients, we have

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad 2\beta = \frac{R}{L} .$$

Putting all these definitions together, we can solve for the current and voltage profiles as a function of frequency

$$\begin{aligned} I(t) &= \frac{V_s \omega}{L} \frac{\cos(\omega t + \delta)}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}} \\ V_R(t) &= I(t)R = V_s 2\beta\omega \frac{\cos(\omega t + \delta)}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}} \\ V_C(t) &= q(t)/C = V_s \omega_0^2 \frac{\sin(\omega t + \delta)}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}} \\ V_L(t) &= -L \frac{d^2q(t)}{dt^2} = V_s \omega^2 \frac{\sin(\omega t + \delta)}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}} \end{aligned}$$

where the phase is given by

$$\tan \delta = \frac{\omega_0^2 - \omega^2}{2\beta\omega} .$$

One useful tool for the study of two equifrequency signals is the XY mode of the oscilloscope. In the standard mode of the oscilloscope, you can think of the display as a standard plot of the signal, with the independent variable, t , on the horizontal axis, and the dependent variable, $V(t)$, on the vertical axis. The XY mode can be thought of as a parametric plot, where the independent variable, t , is implicit (not displayed), while the horizontal and vertical axes trace out two different dependent variables, $V_1(t)$ and $V_2(t)$. The XY mode is most useful when the two signals have commensurate frequencies (their ratio is rational),

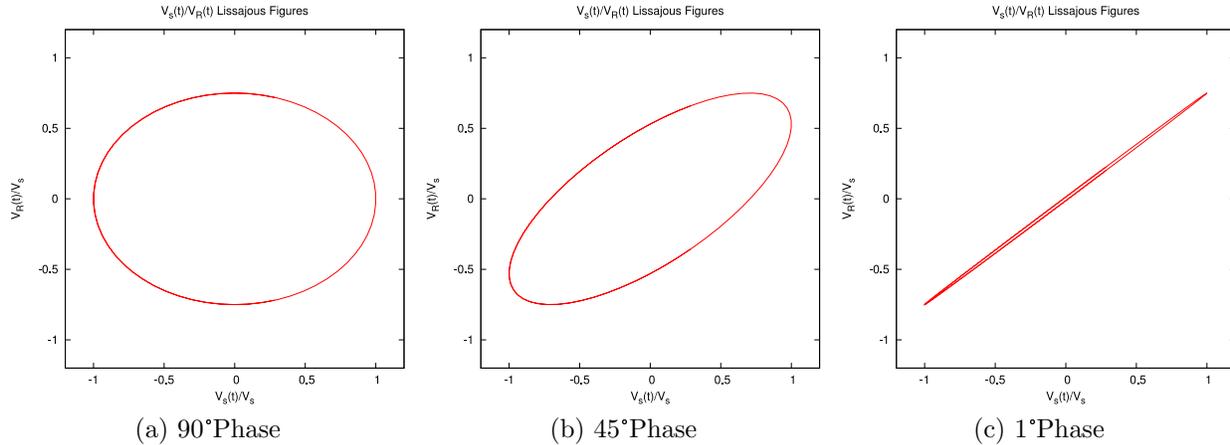


Figure 2: Three example plots for the XY mode of the oscilloscope. In each plot, I show $V_R(t)$ versus $V_s(t)$; the frequencies of the two signals are the same, while the amplitude of $V_R(t)$ is smaller than $V_s(t)$. On the left, they are 90° out of phase; in the middle, 45° ; on the right, they are one degree out of phase.

particularly when their frequencies are the same. In these cases, the XY signals give both relative frequency and relative phase information in an easy to interpret manner; such plots are called *Lissajous Figures*. Figure 2 shows a few of these plots. When the two signals have the same frequency, the figure traces an ellipse. The major axis of the ellipse rotates, and the minor axis shrinks, as the phase changes. As the phase approaches 0, the minor axis also vanishes.

Just as in the RC and RL settings, we can define a circuit impedance by

$$Z^2 = R^2 + (X_L - X_C)^2 ,$$

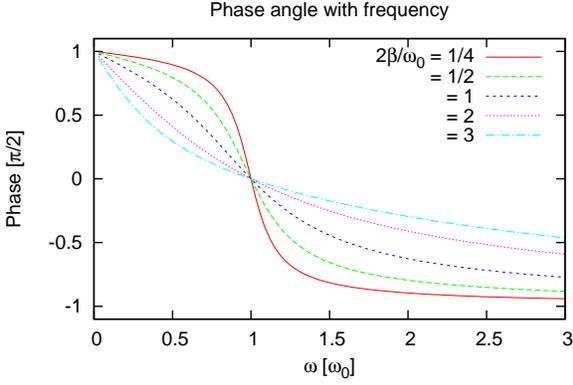
which has all the same consequences for the relationships between the voltage amplitudes as it did for the RC and RL circuits. We could rewrite this equation in terms of the voltage amplitudes (that's mostly left to you; see the Pre-Lab exercises); here we'll only note the phase. You can show

$$\tan \delta = \frac{X_L - X_C}{R} .$$

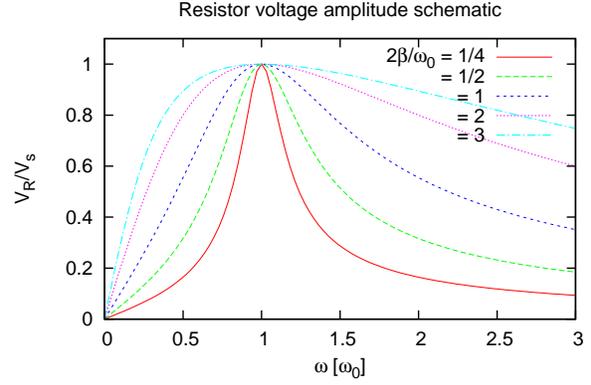
Just as we did for the RC and RL circuits, we should consider the behavior of the RLC circuit as a function of frequency . . . and we're in for some new surprises, with very rich and interesting phenomenology. Consider first the phase. Notice that $\delta = 0$ when $X_L = X_C$. This is called the *phase resonance* of the circuit. At what frequency, ω_R , does this happen?

$$X_L = X_C = \omega_R L = \frac{1}{\omega_R C} \longrightarrow \omega_R = \frac{1}{\sqrt{LC}} = \omega_0 .$$

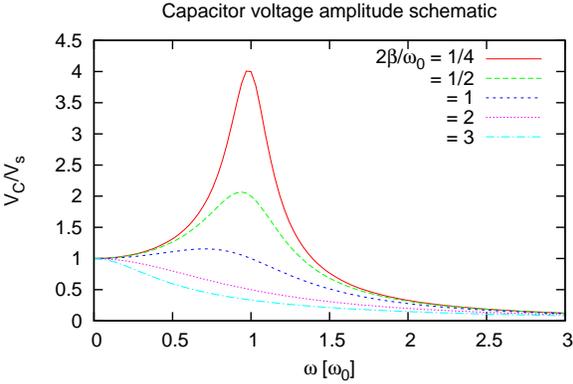
This is called the *natural* frequency of the circuit. Where X_L dominates X_C , the current lags the drive voltages, while the current leads the drive voltage in the opposite case. Thus, the phase can vary between $-\pi/2$ and $\pi/2$, depending on the values of the circuit components.



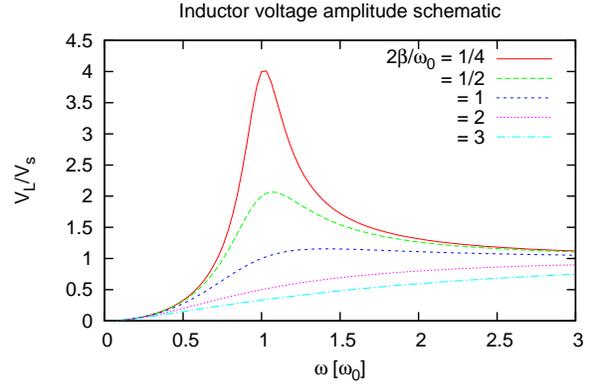
(a) Phases



(b) Resistor voltage amplitude



(c) Capacitor voltage amplitude



(d) Inductor voltage amplitude

Figure 3: The phase angle as a function of angular frequency is on the upper left. The resistor voltage/current amplitudes are displayed on the upper right. The lower left shows the capacitor voltage amplitude, while the lower right shows the inductor voltage amplitude. In all cases, the frequency is normalized in units of $\omega_0 = 1/\sqrt{LC}$. Because the phase and amplitudes are also a function of $2\beta = R/L$, we plot families of curves for various values of $2\beta/\omega_0$. The phase is normalized to $\pi/2$, while the amplitudes are normalized to V_s .

There are a number of other resonances for this circuit. We can, for instance, look for the maxima of the voltage amplitudes, the so called *amplitude resonances* of the circuit; see Figure 3. To predict these, we extremize the amplitudes versus the frequencies ($\frac{dA(\omega)}{d\omega} = 0$). Clearly the current and the voltage across the resistor will be maximized at the same frequency:

$$\frac{dV_R(\omega)}{d\omega} \Big|_{\omega_R} = 0 \longrightarrow \omega_R = \omega_0 ,$$

or the *current amplitude resonance* occurs at the same frequency as the natural oscillation frequency of the circuit. Interestingly, the amplitude resonance for the capacitor and inductor voltages are *not* the same as for the current! For the capacitor,

$$\frac{dV_C(\omega)}{d\omega} \Big|_{\omega_C} = 0 \longrightarrow \omega_C = \sqrt{\omega_0^2 - 1/2(2\beta)^2} ,$$

the resonant voltage amplitude across the capacitor occurs at a *lower* frequency than the phase resonance! For the inductor,

$$\frac{dV_L(\omega)}{d\omega} \Big|_{\omega_L} = 0 \longrightarrow \omega_L = \sqrt{\frac{\omega_0^4}{\omega_0^2 - 1/2(2\beta)^2}} .$$

the resonant voltage amplitude occurs at a frequency *higher* than the phase resonance. Of course, these last two resonance conditions will only occur if the radical is real.

4 Procedures

You should receive two multimeters (one of which should be a BK-5460), an oscilloscope, a function generator, a decade resistance box, a decade capacitance box, and an inductor.

1. First, select component values for testing. Select a frequency between 300 Hz and 600 Hz, and a value for C between $0.06 \mu\text{F}$ and $0.1 \mu\text{F}$. Record the value of the inductance, L . Measure and record the values of the inductor resistance R' , and C . Calculate $X = |X_L - X_C|$ and choose a value for $R + R' \approx 1.2X$. Set, measure and record R .
2. Configure the circuit for testing shown in Figure 1. Insert the Simpson multimeter to record the AC current.
3. Using the BK Precision meter, record the frequency f , and the RMS AC voltages across the signal generator V_s , the resistor V_R , the capacitor V_C , and the inductor V_L . Are these values consistent?
4. Measure the phase shift between the current and applied voltage for your chosen frequency. Connect the oscilloscope so as to measure the voltage across the resistor and signal generator; make sure the negative inputs share a common reference point. Make

sure the two signal baselines are centered with respect to the horizontal and vertical axes of the oscilloscope, and adjust the voltage and time scales so that slightly more than one cycle of both waveforms is visible. Measure the phase shift as you did in the previous lab. Increase the frequency by 50%, and determine the phase shift again. Halve the initial frequency, and repeat.

5. Next, find the resonant frequency. First, predict the value. Then, go searching for it experimentally. There are three methods to do this:
 - (a) Find the frequency which maximizes the current (or the voltage across the resistor),
 - (b) Find the frequency which eliminates the phase shift in AB mode, or
 - (c) Find the frequency which collapses the XY mode Lissajous figure to a line.

In an ideal world each method should give the same result, although the last method is both easiest and most accurate. Do all three methods, in fact, produce the same result?

6. Next, map out the amplitude of the current response. Without changing R or C , vary the frequency over, say, ten points, and record the frequency, RMS voltage V_s and RMS current I at those points. The low frequency should be about half the phase resonance, the high frequency should be about twice the phase resonance, and the middle point should be at the phase resonance.
7. Finally, let's search for the amplitude resonances of $V_C(t)$ and $V_L(t)$. Without changing C , set $R + R' = 0.4X$.² Confirm that the phase resonance has the same frequency. Predict the frequencies of both amplitude resonances. Using either the multimeter or the oscilloscope, search for the amplitude resonances. Do they match your prediction?

A Derivation of Solutions

Applying Kirchoff's rules to the series RLC circuit leads to a second order linear differential equation with constant coefficients, with a driving function. We won't solve this function in all its glory as we did for the first order equation that arises for the RC and RL equations. In fact, solving the equation is somewhat beyond the level of the prerequisite mathematics for this course, so we'll just quote the solution here.³

Let's write the differential equation in the form

$$\frac{d^2q(t)}{dt^2} + 2\beta\frac{dq(t)}{dt} + \omega_0^2q(t) = A \cos \omega t .$$

A full solution to this equation can be written in terms of two functions:

²Why do we have to change R ?

³Although you can look up the solution methods on the net, or in any standard sophomore level physics or mathematics text. I take the notation from Marion and Thornton's *Classical Dynamics*.

1. The “complementary” or “homogeneous” solution $q_c(t)$, when $A = 0$. This is the transient solution, and decays away exponentially; we’ll not dwell on this any further.
2. The “particular” solution $q_p(t)$, where we don’t drop the driving term. We previously called this one the “steady state” solution. Typically, we take “trial solutions” of the same form as the driving term, and see if we can come work up a function that satisfies the equation.

Let’s try a particular solution that’s a sinusoid. Before diving in, let’s think about the *form* of the physical solution that we’re looking for here. In our specific problem, the solution to the differential equation is the charge on the capacitor, but the physics we’re interested in is the phase difference between the *current* and the drive voltage. Thus, we want a current that is of the same functional form as the drive voltage, plus a phase offset. Since we choose a drive voltage that is a cosine, and current is the derivative of the charge, we want to choose a steady state (particular) solution of the form

$$q_p(t) = D \sin(\omega t + \delta) .$$

To show this is a solution, we need to find D and δ , which we do by substituting into the differential equation, to obtain

$$-D\omega^2 \sin(\omega t + \delta) + 2\beta D\omega \cos(\omega t + \delta) + D\omega_0^2 \sin(\omega t + \delta) = A \cos \omega t .$$

Next, we expand the sin and cos terms, since:

$$\cos(a + b) = \cos \omega t \cos \delta - \sin \omega t \sin \delta \quad \sin(a + b) = \sin \omega t \cos \delta + \cos \omega t \sin \delta ,$$

which gives six terms on the left, half of which contain a $\sin \omega t$ and the other have contain $\cos \omega t$. The right hand side contains only $\cos \omega t$. If this is a solution, the coefficients of the $\sin \omega t$ and $\cos \omega t$ terms must separately be equal, and must be collectively consistent. Thus, we have two equations for the two unknowns D and δ

$$\begin{aligned} \sin \omega t : & \quad -D\omega^2 \cos \delta - 2\beta D\omega \sin \delta + D\omega_0^2 \cos \delta = 0 \\ \cos \omega t : & \quad -D\omega^2 \sin \delta + 2\beta D\omega \cos \delta + D\omega_0^2 \sin \delta = A . \end{aligned}$$

The $\sin \omega t$ equation can be solved for δ

$$\tan \delta = \frac{\omega_0^2 - \omega^2}{2\beta\omega} .$$

Now that we have δ , we can use the second equation to solve for D

$$D = \frac{A}{(\omega_0^2 - \omega^2) \sin \delta + 2\beta\omega \cos \delta} .$$

How do we substitute for δ in this latter equation? Using the trigonometric relations

$$\tan \delta = \frac{\sin \delta}{\cos \delta} \quad \sin^2 \delta + \cos^2 \delta = 1 ,$$

you can show that

$$\sin \delta = \frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}} \quad \cos \delta = \frac{2\beta\omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}} .$$

Substituting these values, you will obtain the coefficient D

$$D = \frac{A}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}} .$$

Pre-Lab Exercises

Answer these questions as instructed on Blackboard; make sure to submit them before your lab session!

1. Derive a relationship between the voltage amplitudes V_s , V_R , V_C , and V_L for the RLC circuit. Hint: how is V_s related to the impedance?
2. A series RLC circuit is driven at 500 Hz by a sine wave generator. It has parameters $R = 5 \text{ k}\Omega$, $L = 2 \text{ H}$, and $C = 2 \text{ }\mu\text{F}$. What is the impedance of the circuit?
3. What is the phase resonance frequency for the circuit in the previous question?
4. Does this circuit have amplitude resonances? If so, what are their frequencies?
5. If you use instead a $0.08 \text{ }\mu\text{F}$ capacitor, does the circuit have amplitude resonances? If so, what are their frequencies? What is the new phase resonance?

Post-Lab Exercises

1. From your recorded inductance, and the measured resistance, capacitance, inductance, and initial frequency, determine the impedance of your circuit. Make sure to estimate your uncertainties. Determine the impedance experimentally via another method, taking care of the uncertainties. Do you get the same results?
2. Estimate the uncertainties on the measured values of V_s , V_L , V_C , and V_R . Are these values consistent with each other? Explain what you mean by “consistent”.
3. From your measurements in Step 4 of the procedure, determine the phase shift at each of the three measured frequencies, including an estimate of the uncertainty. How do these compare to the theoretical predictions?
4. Are your three measurements of the phase resonance frequency in Step 5 consistent with each other? With theoretical prediction?
5. Describe qualitatively what happens to your signals when you vary the frequency around the phase resonance.
6. Is your data from Step 6 consistent with the predictions of theory? Specifically, do the voltage and current amplitudes measured by oscilloscope and by multimeter match, within uncertainties, and do they comport with theoretical expectations?
7. Why did you have to change the resistance value for Step 7? Did you find the amplitude resonances? If so, do their values agree with your theoretical predictions?
8. Discuss briefly whether you have met the objectives of the lab exercises.